Third-neighbour and other four-point correlation functions of spin-1/2 *XXZ* chain

2004 J. Phys. A: Math. Gen. 37 5097

(http://iopscience.iop.org/0305-4470/37/19/001)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 5097–5123 PII: S0305-4470(04)76561-0

Third-neighbour and other four-point correlation functions of spin-1/2 *XXZ* chain

Go Kato[1], **Masahiro Shiroishi**[2], **Minoru Takahashi**[2] **and Kazumitsu Sakai**[3]

[1] Department of Physics, Graduate School of Science, University of Tokyo, Hongo 7-3-1, Bunkyo-ku, Tokyo 113-0033, Japan
[2] Institute for Solid State Physics, University of Tokyo, Kashiwanoha, Kashiwa-shi, Chiba 277-8571, Japan
[3] Department of Physics, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo 152-8551, Japan

E-mail: kato@monet.phys.s.u-tokyo.ac.jp, siroisi@issp.u-tokyo.ac.jp, mtaka@issp.u-tokyo.ac.jp and sakai@stat.phys.titech.ac.jp

Received 24 February 2004
Published 27 April 2004
Online at stacks.iop.org/JPhysA/37/5097
DOI: 10.1088/0305-4470/37/19/001

Abstract

The correlation functions of the spin-1/2 *XXZ* chain in the ground state were expressed in the form of multiple integrals for $-1 < \Delta \leqslant 1$ and $1 < \Delta$. In particular, adjacent four-point correlation functions were given as certain four-dimensional integrals. We show that these integrals can be reduced to polynomials with respect to specific one-dimensional integrals. The results give the polynomial representation of the third-neighbour correlation functions.

PACS numbers: 75.10.Jm, 75.50.Ee, 02.30.Ik

1. Introduction

The one-dimensional spin-1/2 *XXZ* model is a quantum integrable system and has been well studied in statistical physics. The Hamiltonian is given by

$$H = \sum_{j=-\infty}^{\infty} \left[S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z \right] \tag{1.1}$$

where $S^\alpha = \sigma^\alpha/2\,(\alpha = x, y, z)$ with σ^α being Pauli matrices. Here Δ is the anisotropy parameter. The eigenvalues and the eigenfunctions of the system were obtained by Bethe ansatz [1, 2], which allowed us to investigate many physical quantities [3]. However, as for the correlation functions, it is not fully understood yet except for the $\Delta = 0$ case, where the system reduces to the free-fermion model [4–7].

For general $\Delta \neq 0$ cases, nevertheless, there is also some recent progress in the evaluation of the correlation functions, especially for the spin–spin correlation functions (SSCF) $\langle S_0^\alpha S_m^\alpha \rangle$ and the emptiness formation probability (EFP) $\left\langle \prod_{j=1}^{m} \left(S_j^z + \frac{1}{2}\right)\right\rangle$ [8]. For example, via the field theoretical approach, we can now discuss the long-distance asymptotic behaviour of the SSCF [9–12], which decays algebraically in the critical region $-1 < \Delta \leqslant 1$. The peculiar Gaussian decay of the EFP has also been studied recently [13, 14]. However, this work contains some assumptions or approximations. Hence, we usually need some other means, such as numerical simulations, to appeal their validity. So it is more desirable if we can evaluate the correlations for an arbitrary m exactly first, and derive its asymptotic behaviour thereafter[4].

In fact, it is already known that the analytical expressions for the SSCF and the EFP at arbitrary m as well as other correlations among adjacent m-sites are given by certain m-dimensional integrals [15–21]. Unfortunately, there is a serious problem for these multiple integrals that it is hard to evaluate them, even numerically, because of the complicated structures of the integrand. For this reason, it has been an important and challenging problem to simplify these multiple integral expressions systematically. Actually, a few years ago, Boos *et al* succeeded in simplifying the multiple integrals for EFP at $\Delta = 1$. They could express the EFP up to $m = 6$ as polynomials with respect to the specific values of ζ function [22–25]. The same method was applied to the third-neighbour correlations at $\Delta = 1$ [26]. Furthermore, we have recently succeeded in generalizing the method to the general anisotropy parameter Δ. In particular we have reported in our previous papers that the three-dimensional integrals relevant for the second-neighbour correlations can be simplified to one-dimensional ones [27, 28]. The main purpose of this paper is to present a detailed account of our method. Further, applying our method, we evaluate all the adjacent four-point correlations including the third-neighbour correlation functions for general Δ. Our significant finding is that these correlation functions are expressed as polynomials with respect to the following specific one-dimensional integrals,

$$\zeta_\eta(j) := \int_{C_-} \mathrm{d}x \frac{1}{\sinh x} \frac{\cosh \eta x}{\sinh^j \eta x} \tag{1.2}$$

$$\zeta'_\eta(j) := \int_{C_-} \mathrm{d}x \frac{1}{\sinh x} \frac{\partial}{\partial \eta} \frac{\cosh \eta x}{\sinh^j \eta x} \tag{1.3}$$

where the integration path is given by $C_- = (-\infty - \pi\mathrm{i}/2, \infty - \pi\mathrm{i}/2)$ and the anisotropy parameter η $(\Delta = \cos \pi\eta)$ is either a real number from 0 to 1 or a purely imaginary number. This property will probably be valid for any correlation functions for the *XXZ* chain (cf [29]). We also remark that the one-dimensional integrals (1.2) and (1.3) can be integrated analytically when η takes a real rational number. In such a case, we can obtain the complete analytic values for the correlation functions.

The outline of this paper is as follows. In section 2, we show a general strategy to evaluate the multiple integral in the massless case. In section 3, we point out the similarity of the procedures in the massless case and the massive case. In section 4, we evaluate, as an example, an adjacent two-point correlation function by the strategy explained in sections 2 and 3. This derivation does not depend on Δ, i.e. it works both in the massive and the massless cases. In section 5, we show the polynomial representation for the third-neighbour correlation functions and some of their analytical values at special points of Δ. The last section is devoted to the concluding remarks. The outline of the derivation for an

[4] Note that such a programme has been achieved for the asymptotic behaviour of the EFP $P(m)$ at $\Delta = 1/2$.

adjacent four-point correlation function is presented in appendix A. In appendix B, we list the polynomial expressions of all the independent correlation functions among up to four adjacent points.

2. General discussion in the massless case

In the massless region $-1 < \Delta \leqslant 1$ the correlation functions $F\left[\begin{smallmatrix}\epsilon'_1 \cdots \epsilon'_n \\ \epsilon_1 \cdots \epsilon_n\end{smallmatrix}\right]$ among adjacent n-sites are given by the following multiple integrals [21],

$$
\begin{aligned}
F\begin{bmatrix}\epsilon'_1 & \cdots & \epsilon'_n \\ \epsilon_1 & \cdots & \epsilon_n\end{bmatrix} &:= \left\langle E^{(1)}_{\epsilon'_1\epsilon_1} \cdots E^{(n)}_{\epsilon'_n\epsilon_n} \right\rangle \\
&= \nu^{-n(n-1)/2} \prod_{a\in A_+} \int_{C_+} \frac{-\mathrm{d}x_a}{2\pi \mathrm{i}} \prod_{a\in A_-} \int_{C_-} \frac{\mathrm{d}x_a}{2\pi \mathrm{i}} \prod_{1\leqslant l<l'\leqslant n} \frac{\sinh(x_l - x_{l'})}{\sinh \nu(x_l - x_{l'} - \pi \mathrm{i})} \\
&\quad \times \prod_{a\in A_+} \frac{\sinh^{\bar{a}-1} \nu x_a \cdot \sinh^{n-\bar{a}} \nu(x_a - \pi \mathrm{i})}{\sinh^n x_a} \\
&\quad \times \prod_{a\in A_-} \frac{\sinh^{\bar{a}-1} \nu x_a \cdot \sinh^{n-\bar{a}} \nu(x_a + \pi \mathrm{i})}{\sinh^n x_a}. \qquad (2.1)
\end{aligned}
$$

Here ϵ and ϵ' take either + or −, and $E_{\epsilon'\epsilon}$ are the 2×2 matrices with 1 at the (ϵ', ϵ) element and 0 elsewhere. $E^{(j)}_{\epsilon'\epsilon}$ denote the operators acting as $E_{\epsilon'\epsilon}$ on the jth spin site and as the identity on the other sites. Then the correlation function $\left\langle E^{(1)}_{\epsilon'_1\epsilon_1} \cdots E^{(1)}_{\epsilon'_n\epsilon_n}\right\rangle$ is defined by the expected value of the operator $E^{(1)}_{\epsilon'_1\epsilon_1} \cdots E^{(n)}_{\epsilon'_n\epsilon_n}$ with respect to the ground state of the *XXZ* Hamiltonian (1.1). For the description of the multiple integrals, we use the same notation as [21]. Let us note this briefly below. First the anisotropy parameter ν is defined by the relations $\Delta = \cos \pi\nu$. As we are considering the massless region $-1 < \Delta \leqslant 1$ here, the parameter takes $0 \leqslant \nu < 1$. The contours $C_\pm$ in each integral go from $\pm\pi \mathrm{i}/2 - \infty$ to $\pm\pi \mathrm{i}/2 + \infty$. The bars, e.g. $\bar{a}$, denote the mapping from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$ defined by the following condition. The +s in the sequence $-\epsilon'_1, \ldots, -\epsilon'_n, \epsilon_n, \ldots, \epsilon_1$ are $-\epsilon'_{\bar{1}}, \ldots, -\epsilon'_{\bar{s}}, \epsilon_{\overline{s+1}}, \ldots, \epsilon_{\bar{n}}$ where s is the number of − in the sequence $\epsilon'_1, \ldots, \epsilon'_n$. The sets A_+ and A_- are defined as $\{j|\epsilon'_{\bar{j}} = -\}$ and $\{j|\epsilon_{\bar{j}} = +\}$ respectively. Note that here we assume $\#A_+ + \#A_- = n$, because otherwise the expected value $F\left[\begin{smallmatrix}\epsilon'_1 \cdots \epsilon'_n \\ \epsilon_1 \cdots \epsilon_n\end{smallmatrix}\right]$ is equal to zero.

Let us describe a strategy that may be used for the evaluation of general $F\left[\begin{smallmatrix}\epsilon'_1 \cdots \epsilon'_n \\ \epsilon_1 \cdots \epsilon_n\end{smallmatrix}\right]$. We first represent the integral formula (2.1) as follows,

$$
F\begin{bmatrix}\epsilon'_1 & \cdots & \epsilon'_n \\ \epsilon_1 & \cdots & \epsilon_n\end{bmatrix} = \nu^{-n(n-1)/2} \prod_{j=1}^{n} \int_{C_-} \frac{\mathrm{d}x_j}{2\pi \mathrm{i}} U(x_1, \ldots, x_n) T(\exp(2x_1\nu), \ldots, \exp(2x_n\nu)) \qquad (2.2)
$$

where

$$
U(x_1, \ldots, x_n) = \frac{\prod_{1\leqslant l<l'\leqslant n} \sinh(x_l - x_{l'})}{\prod_{l=1}^{n} \sinh^n x_l} \qquad (2.3)
$$

$$T(q^{-1}y_1,\ldots,q^{-1}y_s,y_{s+1},\ldots,y_n) = \frac{\prod_{a=A_+}(y_a-1)^{\bar{a}-1}\left(q^{-\frac{1}{2}}y_a-q^{\frac{1}{2}}\right)^{n-\bar{a}}\prod_{a=A_-}(y_a-1)^{\bar{a}-1}\left(q^{\frac{1}{2}}y_a-q^{-\frac{1}{2}}\right)^{n-\bar{a}}}{2^{n(n-1)/2}\prod_{1\leqslant l,l'\leqslant n}\left(q^{-\frac{1}{2}}y_l-q^{\frac{1}{2}}y_{l'}\right)} \tag{2.4}$$

and

$$q \equiv \exp(2\pi\mathrm{i}\nu). \tag{2.5}$$

We can make a lot of simplification without evaluating integration but using only some simple observations and properties of the integrand of (2.2). For this purpose, let us first define a 'weak' equality $\sim$. We call that two functions $F(y_1,\ldots,y_n)$ and $G(y_1,\ldots,y_n)$ are 'weakly' equivalent

$$F(y_1,\ldots,y_n)\sim G(y_1,\ldots,y_n) \tag{2.6}$$

if

$$\prod_{j=1}^{n}\int_{C_-}\frac{\mathrm{d}x_j}{2\pi\mathrm{i}}U(x_1,\ldots,x_n)[F(\exp(2x_1\nu),\ldots,\exp(2x_n\nu)) - G(\exp(2x_1\nu),\ldots,\exp(2x_n\nu))]=0. \tag{2.7}$$

Let us also introduce a 'canonical' form of the function by the following formula,

$$T^c(y_1,\ldots,y_n)=\sum_{j=0}^{[\frac{n}{2}]}P_j^{(n)}\prod_{k=1}^{j}\frac{1}{y_{2k}-y_{2k-1}} \tag{2.8}$$

where $P_j^{(n)}$ are some polynomials with respect to y_k such that

$$P_j^{(n)}\equiv P_j^{(n)}(y_1,y_3,\ldots,y_{2j-1}|y_{2j+1},y_{2j+2},\ldots,y_n) = \sum_{\substack{0\leqslant i_1\leqslant i_3\leqslant\cdots\leqslant i_{2j-1}\leqslant n-1\\ 0\leqslant i_{2j+1}<i_{2j+2}<\cdots<i_n\leqslant n-1}} C_{i_1,i_3,\ldots,i_{2j-1}}^{i_{2j+1},i_{2j+2},\ldots,i_n}y_1^{i_1}\cdots y_n^{i_n} \tag{2.9}$$

and the coefficients $C_{i_1,\ldots,i_{2j-1}}^{i_{2j+1},\ldots,i_n}$ are rational functions of $q^{1/2}$. Our hypothesis is that for any n and ϵ_m one can reduce the function T defined by (2.4) to the canonical form, i.e. there exist polynomials in (2.8) such that

$$T(y_1,\ldots,y_n)\sim T^c(y_1,\ldots,y_n). \tag{2.10}$$

Unfortunately, for the moment we do not have a proof of this statement for any n but we demonstrate how it works for $n=2,4$ in the following sections.

In fact, the problem of the evaluation of $F\left[\begin{smallmatrix}\epsilon_1'\cdots\epsilon_n'\\ \epsilon_1\cdots\epsilon_n\end{smallmatrix}\right]$ given by the integral (2.2) can be reduced to two steps. The first step corresponds to the transformation into the 'canonical' form. The second step is the integration by means of this 'canonical' form. For these procedures, we need several relations as follows.

Proposition 1. *Since the function $U(x_1,\ldots,x_n)$ is antisymmetric with respect to transposition of any pair of integration variables x_j and x_k, we have*

$$\prod_{j=1}^{n}\int_{C_-}\frac{\mathrm{d}x_j}{2\pi\mathrm{i}}U(x_1,\ldots,x_n)S(\exp(2x_1\nu),\ldots,\exp(2x_n\nu))=0 \tag{2.11}$$

if the function S is symmetric for at least one pair of x_s. Therefore for an arbitrary function $f(y_1,\ldots,y_n)$, we can transpose any pair of y_s taking into consideration appearance of additional sign due to the antisymmetry of $U(x_1,\ldots,x_n)$. For example, by transposing y_j with y_k, we get

$$f(y_1,\ldots,y_j,\ldots,y_k,\ldots,y_n) \sim -f(y_1,\ldots,y_k,\ldots,y_j,\ldots,y_n). \tag{2.12}$$

Proposition 2. *Let the rational function $f(y_1,\ldots,y_n)$ have poles of the terms only $1/(y_j - q^a y_k)$ and $1\big/y_j^a$ where a is an integer, i.e. the product $U(x_1,\ldots,x_n)$ $f(\exp(2x_1\nu),\ldots,\exp(2x_n\nu))$ does not have poles of the terms $1/(\exp(2x_j\nu)-q^a\exp(2x_k\nu))$. Then*

$$(y_j-1)^m f(y_1,\ldots,y_j,\ldots,y_n) \sim -(qy_j-1)^m f(y_1,\ldots,qy_j,\ldots,y_n) \tag{2.13}$$

where m is an integer and $m \geqslant n$.

We can get four useful corollaries from propositions 1 and 2.

Corollary 1.

$$\frac{y_k^a y_l^b}{(y_j-y_k)(y_j-y_l)} g(y_1,\ldots,\hat{y}_k,\ldots,\hat{y}_l,\ldots,y_n) \sim \frac{(y_k y_l)^b \sum_{m=0}^{a-b-1} y_k^m y_l^{a-b-1-m}}{(y_j-y_k)} g(y_1,\ldots,\hat{y}_k,\ldots,\hat{y}_l,\ldots,y_n) \tag{2.14}$$

where $a > b$ and the function $g(y_1,\ldots,\hat{y}_k,\ldots,\hat{y}_l,\ldots,y_n)$ does not depend on y_k and y_l. Relation (2.14) also holds in the case where y_j is replaced with qy_j or $q^{-1}y_j$.

Proof.

$$\begin{aligned}
&\frac{(y_k y_l)^b \sum_{m=0}^{a-b-1} y_k^m y_l^{a-b-1-m}}{(y_j-y_k)} g(y_1,\ldots,\hat{y}_k,\ldots,\hat{y}_l,\ldots,y_n)\\
&\quad\sim \frac{(y_k y_l)^b \sum_{m=0}^{a-b-1} y_k^m y_l^{a-b-1-m}}{2(y_j-y_k)} g(y_1,\ldots,\hat{y}_k,\ldots,\hat{y}_l,\ldots,y_n)\\
&\qquad - \frac{(y_k y_l)^b \sum_{m=0}^{a-b-1} y_k^m y_l^{a-b-1-m}}{2(y_j-y_l)} g(y_1,\ldots,\hat{y}_k,\ldots,\hat{y}_l,\ldots,y_n)\\
&\quad= \frac{y_k^a y_l^b - y_l^a y_k^b}{2(y_j-y_k)(y_j-y_l)} g(y_1,\ldots,\hat{y}_k,\ldots,\hat{y}_l,\ldots,y_n)\\
&\quad\sim \frac{y_k^a y_l^b}{(y_j-y_k)(y_j-y_l)} g(y_1,\ldots,\hat{y}_k,\ldots,\hat{y}_l,\ldots,y_n).
\end{aligned} \tag{2.15}$$

In the two weak equalities above, we have used proposition 1. □

Corollary 2.

$$\begin{aligned}
&\frac{y_j^a y_k^b y_l^c y_m^a}{(y_j-y_k)(y_k-y_l)(y_l-y_m)} g(y_1,\ldots,\hat{y}_j,\ldots,\hat{y}_k,\ldots,\hat{y}_l,\ldots,\hat{y}_m,\ldots,y_n)\\
&\quad\sim \frac{y_j^a y_m^a (y_k y_l)^c \sum_{m=0}^{b-c-1} y_k^m y_l^{b-c-1-m}}{2(y_j-y_k)(y_l-y_m)}\\
&\qquad\times g(y_1,\ldots,\hat{y}_j,\ldots,\hat{y}_k,\ldots,\hat{y}_l,\ldots,\hat{y}_m,\ldots,y_n)
\end{aligned} \tag{2.16}$$

where $b > c$ and the function $g(y_1, \ldots, \hat{y}_j, \ldots, \hat{y}_k, \ldots, \hat{y}_l, \ldots, \hat{y}_m, \ldots, y_n)$ does not depend on y_j, y_k, y_l and y_m. Relation (2.16) is proved as follows:

$$\frac{y_j^a y_k^b y_l^c y_m^a}{(y_j - y_k)(y_k - y_l)(y_l - y_m)} g(y_1, \ldots, \hat{y}_j, \ldots, \hat{y}_k, \ldots, \hat{y}_l, \ldots, \hat{y}_m, \ldots, y_n)$$
$$\sim \frac{y_j^a \left(y_k^b y_l^c - y_l^b y_k^c\right) y_m^a}{2(y_j - y_k)(y_k - y_l)(y_l - y_m)} g(y_1, \ldots, \hat{y}_j, \ldots, \hat{y}_k, \ldots, \hat{y}_l, \ldots, \hat{y}_m, \ldots, y_n)$$
$$= \frac{y_j^a y_m^a (y_k y_l)^c \sum_{m=0}^{b-c-1} y_k^m y_l^{b-c-1-m}}{2(y_j - y_k)(y_l - y_m)} g(y_1, \ldots, \hat{y}_j, \ldots, \hat{y}_k, \ldots, \hat{y}_l, \ldots, \hat{y}_m, \ldots, y_n). \tag{2.17}$$

Here we used proposition 1 similarly as in the weak equality.

Corollary 3.

$$y_j^{l+p} g(y_1, \ldots, \hat{y}_j, \ldots, y_n) \sim - \sum_{k=0, \neq l}^{m} \frac{a_k}{a_l} y_j^{k+p} g(y_1, \ldots, \hat{y}_j, \ldots, y_n) \tag{2.18}$$

where

$$a_k = (-)^k \frac{m!}{k!(m-k)!} (1 + q^{k+p}) \tag{2.19}$$

p, l are integers and the function $g(y_1, \ldots, \hat{y}_j, \ldots, y_n)$ does not depend on y_j and as above it is implied that $m \geqslant n$. In our calculation, corollary 3 is used only in the case $l = 0$ or m.

Proof. Relation (2.18) is derived easily from the relation $\left((y_j - 1)^m y_j^p + (q y_j - 1)^m (q y_j)^p\right)$ $g(y_1, \ldots, \hat{y}_j, \ldots, y_n) \sim 0$, i.e. proposition 2. □

Corollary 4.

$$\frac{y_j^{s+p}}{y_k - y_j} g(y_1, \ldots, \hat{y}_k, \ldots, \hat{y}_j, \ldots, y_n)$$
$$\sim \left(- \sum_{l=0, \neq s}^{m} \frac{b_l}{b_s} \frac{y_j^{l+p}}{y_k - y_j} + \sum_{l=0}^{m} \frac{c_l}{b_s} \sum_{r=0}^{l+p-1} y_k^r (q y_j)^{l+p-r-1} \right) g(y_1, \ldots, \hat{y}_k, \ldots, \hat{y}_j, \ldots, y_n) \tag{2.20}$$

where

$$b_l = (-)^{m-l} \frac{m!}{l!(m-l)!} (1 - q^{l+p-1}) \qquad c_l = (-)^{m-l} \frac{m!}{l!(m-l)!} \tag{2.21}$$

$m \geqslant n$, $s + p \neq 1$ and the function $g(y_1, \ldots, \hat{y}_k, \ldots, \hat{y}_j, \ldots, y_n)$ does not depend on y_k and y_j. In our calculation, corollary 4 is used only in the case $s = 0$ or m.

Proof. We get

$$\frac{(y_j - 1)^m y_j^p}{y_k - y_j} g(y_1, \ldots, \hat{y}_k, \ldots, \hat{y}_j, \ldots, y_n)$$
$$\sim - \frac{(q y_j - 1)^m (q y_j)^p}{y_k - q y_j} g(y_1, \ldots, \hat{y}_k, \ldots, \hat{y}_j, \ldots, y_n)$$
$$= \left(- \frac{(y_k - 1)^m y_k^p}{y_k - q y_j} + \frac{(y_k - 1)^m y_k^p - (q y_j - 1)^m (q y_j)^p}{y_k - q y_j} \right)$$

$$
\begin{aligned}
&\times g(y_1, \dots, \hat{y}_k, \dots, \hat{y}_j, \dots, y_n) \\
&\sim \left(\frac{(qy_k - 1)^m (qy_k)^p}{q(y_k - y_j)} + \frac{(y_k - 1)^m y_k^p - (qy_j - 1)^m (qy_j)^p}{y_k - qy_j} \right) \\
&\times g(y_1, \dots, \hat{y}_k, \dots, \hat{y}_j, \dots, y_n) \\
&\sim \left(\frac{(qy_j - 1)^m (qy_j)^p}{q(y_k - y_j)} + \frac{(y_k - 1)^m y_k^p - (qy_j - 1)^m (qy_j)^p}{y_k - qy_j} \right) \\
&\times g(y_1, \dots, \hat{y}_k, \dots, \hat{y}_j, \dots, y_n).
\end{aligned} \tag{2.22}
$$

Proposition 1 was used in the last weak equality, and proposition 2 was used in the other weak equalities. Therefore,

$$
\begin{aligned}
&\frac{(y_j - 1)^m y_j^p - q^{p-1}(qy_j - 1)^m y_j^p}{y_k - y_j} g(y_1, \dots, \hat{y}_k, \dots, \hat{y}_j, \dots, y_n) \\
&\quad \sim \frac{(y_k - 1)^m y_k^p - (qy_j - 1)^m (qy_j)^p}{y_k - qy_j} g(y_1, \dots, \hat{y}_k, \dots, \hat{y}_j, \dots, y_n)
\end{aligned} \tag{2.23}
$$

holds. Then, expanding both numerators according to the formulae,

$$
(y_j - 1)^m y_j^p - q^{p-1}(qy_j - 1)^m y_j^p = \sum_{l=0}^{m} b_l y_j^{l+p} \tag{2.24}
$$

$$
(y_k - 1)^m y_k^p - (qy_j - 1)^m (qy_j)^p = (y_k - qy_j) \sum_{l=0}^{m} c_l \sum_{r=0}^{l+r-1} y_k^r (qy_j)^{l+p-r-1} \tag{2.25}
$$

we arrive at formula (2.20). □

The relations we wrote above are used to derive the 'canonical' form of (2.10).

Next, we show two elementary integral formulae:

- The first integral formula is

$$
\int_{C_-} \frac{e^{ax}}{\sinh^m x} \, dx = -\frac{2\pi i \prod_{j=1}^{m-1} (a - m + 2j)}{(m-1)!(\exp((a+m)\pi i) - 1)} \tag{2.26}
$$

  where m is an integer and $-m < \mathrm{Re}(a) < m$.
- The second integral formula is

$$
\begin{aligned}
&\int_{C_-} \frac{e^{ax}}{\sinh^m (x + y/2) \sinh^m (x - y/2)} \, dx \\
&\quad = -\frac{2\pi i 2^{2m-1}}{(m-1)!(\exp(a\pi i) - 1)} \left(\frac{\partial^{m-1}}{\partial X^{m-1}} \frac{X^{a/2+m-1}}{(X - e^y)^m} \bigg|_{X = e^{-2y}} \right. \\
&\quad \left. + \frac{\partial^{m-1}}{\partial X^{m-1}} \frac{X^{a/2+m-1}}{(X - e^{-y})^m} \bigg|_{X = e^{2y}} \right)
\end{aligned} \tag{2.27}
$$

  where m is an integer, y is a real number and $-m < \mathrm{Re}(a) < m$. Note that the lhs of (2.27) can be evaluated explicitly when m is given explicitly. For example, we have

$$
\begin{aligned}
&\int_{C_-} \frac{e^{ax}}{\sinh^2 (x + y/2) \sinh^2 (x - y/2)} \, dx \\
&\quad = \frac{\pi i}{\exp(a\pi i) - 1} \left(-\frac{4a \cosh(ya/2)}{\sinh^2 y} + \frac{8 \cosh y \sinh(ya/2)}{\sinh^3 y} \right)
\end{aligned} \tag{2.28}
$$

in the case $m = 2$ and

$$\int_{C_-} \frac{e^{ax}}{\sinh^3(x+y/2)\sinh^3(x-y/2)}\,dx = \frac{\pi i}{\exp(a\pi i)-1}\left(-\frac{2(a^2+8)\sinh(ya/2)}{\sinh^3 y} + \frac{12a\cosh y\cosh(ya/2)}{\sinh^4 y} - \frac{24\sinh(ya/2)}{\sinh^5 y}\right) \tag{2.29}$$

in the case $m = 3$, etc.

Now let us consider the integrals of a special form, namely, the integral

$$\prod_{j=1}^{n}\int_{C_-}\frac{dx_j}{2\pi i}U(x_1,\ldots,x_n) \times P_k^{(n)}(\exp(2x_1\nu),\exp(2x_3\nu),\ldots,\exp(2x_{2k-1}\nu)|\exp(2x_{2k+1}\nu),\exp(2x_{2k+2}\nu),\ldots,\exp(2x_n\nu)) \times\prod_{l=1}^{k}\frac{1}{\exp(2x_{2l})-\exp(2x_{2l-1})} \tag{2.30}$$

where $P_k^{(n)}$ is a polynomial of the form (2.9). Using relations (2.26) and (2.27), we can integrate this expression $(n-k)$ times, namely with respect to $(x_{2j-1}+x_{2j})/2$ and $x_{j'}$ where $j \leqslant k$, $j' > 2k$, while the variables $x_{2j-1} - x_{2j}$ are fixed. In this way, we can evaluate the n-dimensional integral

$$\prod_{j=1}^{n}\int_{C_-}\frac{dx_j}{2\pi i}U(x_1,\ldots,x_n)T^c(\exp(2x_1\nu),\ldots,\exp(2x_n\nu)) \tag{2.31}$$

where U is defined in (2.3) and T^c is a canonical form function (2.8). It results into a polynomial with respect to one-dimensional integrals where the coefficients are rational functions of $\sin\pi\nu$ and $\cos\pi\nu$.

3. General discussion at the massive case

The correlation functions at massive case were obtained by Jimbo *et al* [18, 19]. Here we use the representation in [17],

$$F\begin{bmatrix}\epsilon_1' & \ldots & \epsilon_n'\\ \epsilon_1 & \ldots & \epsilon_n\end{bmatrix} := \left\langle E^{(1)}_{\epsilon_1'\epsilon_1}\ldots E^{(n)}_{\epsilon_n'\epsilon_n}\right\rangle = (\phi i)^{-n(n-1)/2}\prod_{j=1}^{n}\int_{C'}\frac{dx_j}{2\pi i}W(x_1,\ldots,x_n)T(\exp(2x_1\phi i),\ldots,\exp(2x_n\phi i)) \tag{3.1}$$

where C' is the integral path from $-\frac{\pi}{2}(\phi^{-1}+i)$ to $\frac{\pi}{2}(\phi^{-1}-i)$, $T(y_1,\ldots,y_n)$ is defined in (2.10),

$$W(x_1,\ldots,x_n) = A_m\vartheta_2\left(\prod_{m=1}^{n}z_m, q^{1/2}\right)\frac{\prod_{1\leqslant l<l'\leqslant n}\vartheta_1\left(\frac{z_l}{z_{l'}}, q^{1/2}\right)}{\prod_{l=1}^{n}\vartheta_1^n(z_l, q^{1/2})}$$
$$z_m \equiv \exp(\phi x_m i) \quad q = \exp(-\pi\phi) \tag{3.2}$$

$$\vartheta_1(z,p) \equiv -ip^{1/4}(z-z^{-1})\prod_{m=1}^{\infty}(1-p^{2m}z^2)(1-p^{2m}z^{-2})(1-p^{2m}) \tag{3.3}$$

$$\vartheta_2(z, p) \equiv p^{1/4}(z + z^{-1}) \prod_{m=1}^{\infty} (1 + p^{2m} z^2)(1 + p^{2m} z^{-2})(1 - p^{2m}) \tag{3.4}$$

and

$$A_m \equiv \phi^{n(n+1)/2} \prod_{m=1}^{\infty} \left(\frac{1 - q^m}{1 + q^m}\right)^2 \left[2q^{1/8} \prod_{m=1}^{\infty} (1 - q^m)^3\right]^{\frac{m(m+1)}{2} - 1}. \tag{3.5}$$

The parameter ϕ is defined by the relations $\Delta = \cosh \pi\phi$ with $0 \leqslant \phi$.

As in the massless case, we define weak equality $\sim$. Let us state that two functions $F(y_1, \ldots, y_n)$ and $G(y_1, \ldots, y_n)$ are weakly equivalent,

$$F(y_1, \ldots, y_n) \sim G(y_1, \ldots, y_n) \tag{3.6}$$

if

$$\prod_{j=1}^{n} \int_{C'} \frac{\mathrm{d}x_j}{2\pi \mathrm{i}} W(x_1, \ldots, x_n)[F(\exp(2x_1\nu), \ldots, \exp(2x_n\nu)) - G(\exp(2x_1\nu), \ldots, \exp(2x_n\nu))] = 0. \tag{3.7}$$

Moreover, we use the term 'canonical' in the same way as in the massless case (2.8).

Below we show that the correlation function $F\left[{}^{\epsilon'_1 \ldots \epsilon'_n}_{\epsilon_1 \ldots \epsilon_n}\right]$ for the massive case can be evaluated in the same way as the massless case. That is to say, we first obtain the canonical form, and then, reduce the multiple integral into the one-dimensional ones using this canonical form.

In fact, all the equations in the massless case, i.e. (2.12)–(2.25), also hold in the massive case. The reason is that the function $W(x_1, \ldots, x_n)$ has the following properties:

$$W(x_1, \ldots, x_j, \ldots, x_k, \ldots, x_n) = -W(x_1, \ldots, x_k, \ldots, x_j, \ldots, x_n) \tag{3.8}$$

$$W(x_1, \ldots, x_j, \ldots, x_n) = -W(x_1, \ldots, x_j + \pi \mathrm{i}, \ldots, x_n). \tag{3.9}$$

$W(x_1, \ldots, x_n)$ is an elliptic function with periods $\pi\mathrm{i}$ and π/ϕ with respect to each variable x_j. $W(x_1, \ldots, x_n)$ has zero points at $x_j = x_k$ and a pole only at $x_j = 0$, where $j \neq m$. The order of the pole is n. Therefore, the canonical forms for the correlation function in the massive case are the same as those for the massless case.

Next, we remark that the relation

$$W(x_1, \ldots, x_n) = \sum_{m_1, \ldots, m_n = -\infty}^{\infty} U\left(x_1 + \frac{\pi m_1}{\phi}, \ldots, x_n + \frac{\pi m_n}{\phi}\right) \tag{3.10}$$

holds. The proof is as follows. The both sides of (3.10) have the following properties. First, they are elliptic functions with periods π/ϕ and $2\pi\mathrm{i}$ with respect to each variable x_j. Second, the functions have poles only at $x_j = k\pi/\phi + l\pi\mathrm{i}$ and zeros at $x_j = k\pi/\phi + l\pi\mathrm{i} + x_m$, where k and l are arbitrary integers and $j \neq m$. Third, the order of the pole is n and the order of any zero point is 1. Finally, both values

$$\lim_{x_1 \to 0} \lim_{x_2 \to 0} \cdots \lim_{x_n \to 0} W(x_1, \ldots, x_n) \prod_{j=1}^{n} x_j{}^j \tag{3.11}$$

and

$$\lim_{x_1 \to 0} \lim_{x_2 \to 0} \cdots \lim_{x_n \to 0} \sum_{m_1, \ldots, m_n = -\infty}^{\infty} U\left(x_1 + \frac{\pi m_1}{\phi}, \ldots, x_n + \frac{\pi m_n}{\phi}\right) \prod_{j=1}^{n} x_j{}^j \tag{3.12}$$

are 1.

By relation (3.10), we find

$$\prod_{j=1}^{n}\int_{C'}\frac{\mathrm{d}x_j}{2\pi\mathrm{i}}W(x_1,\ldots,x_n)f(\exp(2x_1\phi\mathrm{i}),\ldots,\exp(2x_n\phi\mathrm{i}))$$

$$=\prod_{j=1}^{n}\int_{C_j}\frac{\mathrm{d}x_j}{2\pi\mathrm{i}}U(x_1,\ldots,x_n)f(\exp(2x_1\phi\mathrm{i}),\ldots,\exp(2x_n\phi\mathrm{i})) \tag{3.13}$$

holds, where C_j is the integral path from $-\infty-(1/2-j\delta)\pi\mathrm{i}$ to $\infty-(1/2-j\delta)\pi\mathrm{i}$ and the both integrands have no pole on the integral paths respectively. Then, using relations (2.26) and (2.27), one can conclude that the multiple-integral

$$\prod_{j=1}^{n}\int_{C'}\frac{\mathrm{d}x_j}{2\pi\mathrm{i}}W(x_1,\ldots,x_n)T^c(\exp(\mathrm{i}2x_1\phi),\ldots,\exp(\mathrm{i}2x_n\phi)) \tag{3.14}$$

where W is defined as (3.2) and T^c is a canonical form, reduces to a polynomial with respect to one-dimensional integral. Note that here the coefficients of the polynomial are rational functions of $\cosh\pi\phi$ and $\sinh\pi\phi$.

4. In the case of $F\left[\begin{smallmatrix}+&+\\+&+\end{smallmatrix}\right]$

We illustrate how the procedures in the previous sections work for the simplest case $F\left[\begin{smallmatrix}+&+\\+&+\end{smallmatrix}\right]$, i.e.,

$$\nu^{-1}\int_{C_-}\frac{\mathrm{d}x_1}{2\pi\mathrm{i}}\int_{C_-}\frac{\mathrm{d}x_2}{2\pi\mathrm{i}}U(x_1,x_2)T(\exp(2x_1\nu),\exp(2x_2\nu)) \tag{4.1}$$

in the case $-1<\Delta\leqslant 1$, and

$$-\mathrm{i}\phi^{-1}\int_{C'}\frac{\mathrm{d}x_1}{2\pi\mathrm{i}}\int_{C'}\frac{\mathrm{d}x_2}{2\pi\mathrm{i}}W(x_1,x_2)T(\exp(2x_1\nu),\exp(2x_2\nu)) \tag{4.2}$$

in the case $1<\Delta$. Here $U(x_1,x_2)$ and $W(x_1,x_2)$ are defined in (2.3) and (3.2) respectively. For $F\left[\begin{smallmatrix}+&+\\+&+\end{smallmatrix}\right]$, it is very simple to perform the first step, namely, to get the 'canonical' form (2.8) because we do not need to reduce a power of denominator. Indeed,

$$T(y_1,y_2)=\frac{(y_1-1)(qy_2-1)}{2(y_1-y_2q)}=\frac{qy_2-1}{2}+\frac{(qy_2-1)^2}{2(y_1-qy_2)}\sim\frac{q}{2}y_2-\frac{(y_2-1)^2}{2(y_1-y_2)} \tag{4.3}$$

where we have used relations (2.12) and (2.13). Then, using relation (2.20) for $m=2$, we have

$$\begin{aligned}-\frac{(y_2-1)^2}{2(y_1-y_2)}&=-\frac{y_2^2}{2(y_1-y_2)}+\frac{y_2}{y_1-y_2}-\frac{1}{2(y_1-y_2)}\\&\sim-\frac{1}{2q(y_1-y_2)}+\frac{y_2}{2}+\frac{y_2}{y_1-y_2}-\frac{1}{2(y_1-y_2)}\\&=\frac{2y_2-1-q^{-1}}{2(y_1-y_2)}+\frac{y_2}{2}.\end{aligned} \tag{4.4}$$

Substituting it into the rhs of (4.3), we get

$$T(y_1,y_2)\sim T^c(y_1,y_2)=\frac{q+1}{2}y_2+\left(y_2-\frac{q+1}{2q}\right)\frac{1}{y_1-y_2} \tag{4.5}$$

and this is the 'canonical' form for T, i.e. the polynomials $P_0^{(2)}$ and $P_1^{(2)}$ at (2.8) are equal to $(q+1)y_2/2$ and $y_2-(1+q^{-1})/2$ respectively.

Let us consider an integral

$$
\begin{aligned}
&\nu^{-1}\int_{C_-}\frac{\mathrm{d}x_1}{2\pi\mathrm{i}}\int_{C_-}\frac{\mathrm{d}x_2}{2\pi\mathrm{i}}\frac{\sinh(x_1-x_2)}{\sinh^2 x_1\sinh^2 x_2}\frac{q+1}{2}\exp(2x_2\nu)\\
&=\frac{q+1}{4(2\pi\mathrm{i})^2\nu}\int_{C_-}\mathrm{d}x_1\int_{C_-}\mathrm{d}x_2\frac{\exp(x_1)\exp((2\nu-1)x_2)-\exp(-x_1)\exp((2\nu+1)x_2)}{\sinh^2 x_1\sinh^2 x_2}\\
&=\frac{q+1}{4(2\pi\mathrm{i})^2\nu}\left[(\pi\mathrm{i})\left(\frac{2\pi\mathrm{i}(2\nu-1)}{q+1}\right)-(-\pi\mathrm{i})\left(\frac{2\pi\mathrm{i}(2\nu+1)}{q+1}\right)\right]\\
&=\frac{1}{2}.
\end{aligned}
\tag{4.6}
$$

This integrand corresponds to $U(x_1,x_2)$ times the first term of (4.5). Here we used the formula (2.26) in the second equality. Next, we evaluate another integral

$$
\begin{aligned}
&\nu^{-1}\int_{C_-}\frac{\mathrm{d}x_1}{2\pi\mathrm{i}}\int_{C_-}\frac{\mathrm{d}x_2}{2\pi\mathrm{i}}\frac{\sinh(x_1-x_2)}{\sinh^2 x_1\sinh^2 x_2}\left(\exp(2x_2\nu)-\frac{q+1}{2q}\right)\frac{1}{\exp(2x_1\nu)-\exp(2x_2\nu)}\\
&\qquad=\frac{1}{2(2\pi\mathrm{i})^2\nu}\int_{-\infty}^{\infty}\mathrm{d}y\frac{\sinh y}{\sinh\nu y}\int_{C_-}\mathrm{d}x\frac{\exp(-\nu y)-\frac{q+1}{2q}\exp(-2\nu x)}{\sinh^2(x+y/2)\sinh^2(x-y/2)}\\
&\qquad=\frac{1}{2\pi^2\nu}\int_{-\infty}^{\infty}\mathrm{d}y\frac{1}{\sinh y}\left(\frac{\exp(\nu y)}{\sinh\nu y}-\frac{\mathrm{i}\pi\nu(q+1)\cosh\nu y}{(q-1)\sinh\nu y}\right)\\
&\qquad\quad+\frac{\cosh y}{\sinh^2 y}\left(\frac{\mathrm{i}\pi(q+1)}{(q-1)}-\frac{y\exp(\nu y)}{\sinh\nu y}\right)\\
&\qquad=\int_{C_-}\mathrm{d}y\frac{1}{\sinh y}\left(-\frac{\cos\pi\nu}{2\pi\sin\pi\nu}\frac{\cosh\nu y}{\sinh\nu y}-\frac{1}{2\pi^2}\frac{\partial}{\partial\nu}\frac{\cosh\nu y}{\sinh\nu y}\right).
\end{aligned}
\tag{4.7}
$$

This integrand corresponds to $U(x_1,x_2)$ times the second term of (4.5). In the second equality, we used the formula (2.28), and in the third equality we used the relation

$$
\int_{C_-}\mathrm{d}y\frac{\cosh y}{\sinh^2 y}f(y)=\int_{C_-}\mathrm{d}y\frac{1}{\sinh y}\frac{\mathrm{d}}{\mathrm{d}y}f(y)
\tag{4.8}
$$

where $\lim_{y\to\pm\infty-\mathrm{i}\pi/2} f(y)/\sinh y=0$.

As for the massive case, by exchanging $U(x_1,x_2)$ and C_- with $W(x_1,x_2)$ and C', we can calculate as

$$
-\mathrm{i}\phi^{-1}\int_{C'}\frac{\mathrm{d}x_1}{2\pi\mathrm{i}}\int_{C'}\frac{\mathrm{d}x_2}{2\pi\mathrm{i}}W(x_1,x_2)\frac{q+1}{2}\exp(2x_2\phi\mathrm{i})=\frac{1}{2}
\tag{4.9}
$$

and

$$
\begin{aligned}
&-\mathrm{i}\phi^{-1}\int_{C'}\frac{\mathrm{d}x_1}{2\pi\mathrm{i}}\int_{C'}\frac{\mathrm{d}x_2}{2\pi\mathrm{i}}W(x_1,x_2)\left(\exp(2x_2\phi\mathrm{i})-\frac{q+1}{2q}\right)\frac{1}{\exp(2x_1\phi\mathrm{i})-\exp(2x_2\phi\mathrm{i})}\\
&\qquad=\int_{C_-}\mathrm{d}y\frac{1}{\sinh y}\left(-\frac{\cos\pi\phi\mathrm{i}}{2\pi\sin\pi\phi\mathrm{i}}\frac{\cosh\phi\mathrm{i}y}{\sinh\phi\mathrm{i}y}-\frac{1}{2\pi^2}\frac{\partial}{\partial\phi\mathrm{i}}\frac{\cosh\phi\mathrm{i}y}{\sinh\phi\mathrm{i}y}\right).
\end{aligned}
\tag{4.10}
$$

Actually, once we use relation (3.13), we see that the rest of the derivation is the same as the massless case. Therefore, the final results of these two cases are very similar. More precisely, the results can be exchanged to each other with the exchange of $\phi\mathrm{i}$ and ν. This property always holds in the evaluation of the other correlation functions.

Gathering all the results, we get the following expression,

$$F\begin{bmatrix}+ & +\\ + & +\end{bmatrix} = \frac{1}{2} - \frac{\cos\pi\eta}{2\pi\sin\pi\eta}\int_{C_-} dy \frac{1}{\sinh y}\frac{\cosh\eta y}{\sinh\eta y} - \frac{1}{2\pi^2}\int_{C_-} dy \frac{1}{\sinh y}\frac{\partial}{\partial\eta}\frac{\cosh\eta y}{\sinh\eta y} \tag{4.11}$$

where η is equal to ν in the case $-1 < \Delta = \cos\pi\nu \leqslant 1$ and is equal to $\phi\mathrm{i}$ in the case $1 \leqslant \Delta = \cos\pi\phi$.

5. The third-neighbour correlation functions and some other correlation functions

Using the method in the previous sections, we have analysed all the correlation functions $F[^{\epsilon'_1,\ldots,\epsilon'_n}_{\epsilon_1,\ldots,\epsilon_n}]$ for $n \leqslant 4$. Actually, the derivations of the canonical form from the multiple integral representations of the correlation functions are very similar to each other. So, in appendix A, we explain the outline of its derivation for $F\begin{bmatrix}+ & + & + & +\\ + & + & + & +\end{bmatrix}$ as an example. Also the rest of our tasks, i.e. the evaluation of the integrals with respect to the canonical forms, are straightforward if one uses relations (2.26), (2.27) and the integration by parts like (4.8). Then, we omit the account of these calculations and give the final result in appendix B together with other independent results up to $n = 4$. The correlation functions which are not given in appendix B explicitly should be derived from the relations,

$$F\begin{bmatrix}\epsilon'_1, \epsilon'_2, \ldots, \epsilon'_n\\ \epsilon_1, \epsilon_2, \ldots, \epsilon_n\end{bmatrix} = F\begin{bmatrix}\epsilon'_1, \epsilon'_2, \ldots, \epsilon'_n, +\\ \epsilon_1, \epsilon_2, \ldots, \epsilon_n, +\end{bmatrix} + F\begin{bmatrix}\epsilon'_1, \epsilon'_2, \ldots, \epsilon'_n, -\\ \epsilon_1, \epsilon_2, \ldots, \epsilon_n, -\end{bmatrix} \tag{5.1}$$

$$F\begin{bmatrix}\epsilon'_1, \epsilon'_2, \ldots, \epsilon'_n\\ \epsilon_1, \epsilon_2, \ldots, \epsilon_n\end{bmatrix} = F\begin{bmatrix}\epsilon'_n, \epsilon'_{n-1}, \ldots, \epsilon'_1\\ \epsilon_n, \epsilon_{n-1}, \ldots, \epsilon_1\end{bmatrix} = F\begin{bmatrix}\epsilon_1, \epsilon_2, \ldots, \epsilon_n\\ \epsilon'_1, \epsilon'_2, \ldots, \epsilon'_n\end{bmatrix} = F\begin{bmatrix}-\epsilon'_1, -\epsilon'_2, \ldots, -\epsilon'_n\\ -\epsilon_1, -\epsilon_2, \ldots, -\epsilon_n\end{bmatrix} \tag{5.2}$$

$$F\begin{bmatrix}\epsilon'_1, \epsilon'_2, \ldots, \epsilon'_n\\ \epsilon_1, \epsilon_2, \ldots, \epsilon_n\end{bmatrix} = 0 \quad \text{in case} \quad \#\{j|\epsilon'_j = +\} \neq \#\{j|\epsilon_j = +\}. \tag{5.3}$$

From our results, we can express many correlation functions as a polynomial with respect to one-dimensional integrals. For example, the third-neighbour correlation functions $\langle S^x_j S^x_{j+3}\rangle$ and $\langle S^z_j S^z_{j+3}\rangle$ are given by

$$\begin{aligned}\langle S^x_j S^x_{j+3}\rangle = {} & \frac{3}{4\pi(1+2c_2)s_1}\zeta_\eta(1) + \frac{c_1(-1+2c_2)}{4\pi^2}\zeta'_\eta(1)\\ & - \frac{(-1+7c_2+4c_4)s_1}{2\pi(1+2c_2)}\zeta_\eta(3) - \frac{c_1{s_1}^2}{2\pi^2}\zeta'_\eta(3) - \frac{5(2c_2+c_4)s_1}{4\pi(1+2c_2)}\zeta_\eta(5)\\ & - \frac{c_1c_2{s_1}^2}{4\pi^2}\zeta'_\eta(5) - \frac{c_1(-12+4c_2-7c_4)}{4\pi^2(1+2c_2)}\zeta_\eta(1)\zeta_\eta(3)\\ & + \frac{(6+3c_2+c_4)s_1}{8\pi^3}\zeta'_\eta(1)\zeta_\eta(3) + \frac{(-1+3c_2)s_1}{8\pi^3}\zeta_\eta(1)\zeta'_\eta(3) + \frac{{c_1}^3{s_1}^2}{4\pi^4}\zeta'_\eta(1)\zeta'_\eta(3)\\ & + \frac{5c_1(9+4c_2+5c_4)}{8\pi^2(1+2c_2)}\zeta_\eta(1)\zeta_\eta(5) + \frac{5{c_1}^2(2+c_2)s_1}{4\pi^3}\zeta'_\eta(1)\zeta_\eta(5)\\ & + \frac{{c_1}^2(2+c_2)s_1}{4\pi^3}\zeta_\eta(1)\zeta'_\eta(5) + \frac{3{c_1}^3{s_1}^2}{4\pi^4}\zeta'_\eta(1)\zeta'_\eta(5) - \frac{3c_1(9+4c_2+5c_4)}{16\pi^2(1+2c_2)}\zeta_\eta(3)^2\\ & - \frac{{c_1}^2(2+c_2)s_1}{4\pi^3}\zeta_\eta(3)\zeta'_\eta(3) - \frac{{c_1}^3{s_1}^2}{8\pi^4}\zeta'_\eta(3)^2\end{aligned} \tag{5.4}$$

$$\langle S^z_j S^z_{j+3}\rangle = \frac{1}{4} - \frac{3c_1(-1+2c_2)}{2\pi(1+2c_2)s_1}\zeta_\eta(1) - \frac{1}{2\pi^2}\zeta'_\eta(1) + \frac{(3+2c_2)(7+c_4)s_1}{4\pi c_1(1+2c_2)}\zeta_\eta(3)$$

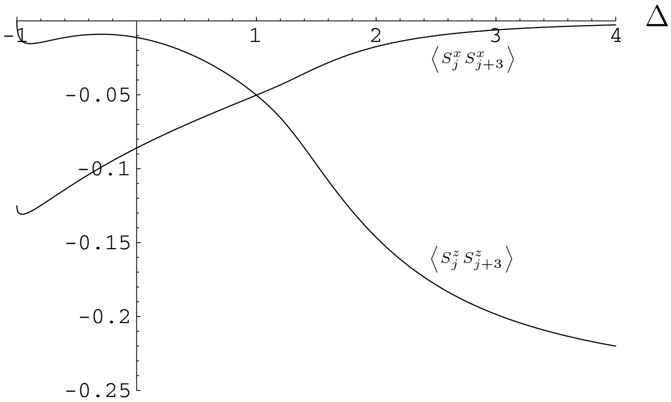


Figure 1. The third-neighbour correlation functions for the *XXZ* chain.

$$
\begin{aligned}
&+\frac{c_1{}^2 s_1{}^2}{\pi^2}\zeta'_\eta(3) + \frac{5(5+3c_2+3c_4+c_6)s_1}{8\pi c_1(1+2c_2)}\zeta_\eta(5) + \frac{c_2 s_1{}^2}{2\pi^2}\zeta'_\eta(5) \\
&- \frac{(2+23c_2+4c_4+c_6)}{4\pi^2(1+2c_2)}\zeta_\eta(1)\zeta_\eta(3) - \frac{c_1(4+c_2)s_1}{2\pi^3}\zeta'_\eta(1)\zeta_\eta(3) \\
&- \frac{c_1 c_2 s_1}{2\pi^3}\zeta_\eta(1)\zeta'_\eta(3) - \frac{c_1{}^2 s_1{}^2}{2\pi^4}\zeta'_\eta(1)\zeta'_\eta(3) - \frac{5(8+23c_2+4c_4+c_6)}{8\pi^2(1+2c_2)}\zeta_\eta(1)\zeta_\eta(5) \\
&- \frac{5c_1(2+c_2)s_1}{2\pi^3}\zeta'_\eta(1)\zeta_\eta(5) - \frac{c_1(2+c_2)s_1}{2\pi^3}\zeta_\eta(1)\zeta'_\eta(5) - \frac{3c_1{}^2 s_1{}^2}{2\pi^4}\zeta'_\eta(1)\zeta'_\eta(5) \\
&+ \frac{3(8+23c_2+4c_4+c_6)}{16\pi^2(1+2c_2)}\zeta_\eta(3)^2 + \frac{c_1(2+c_2)s_1}{2\pi^3}\zeta_\eta(3)\zeta'_\eta(3) + \frac{c_1{}^2 s_1{}^2}{4\pi^4}\zeta'_\eta(3)^2.
\end{aligned} \tag{5.5}
$$

Here we used the following notation to make the results short,

$$c_j := \cos \pi j \eta \tag{5.6}$$

$$s_j := \sin \pi j \eta \tag{5.7}$$

$$\zeta_\eta(j) := \int_{C_-} \mathrm{d}x \frac{1}{\sinh x} \frac{\cosh \eta x}{\sinh^j \eta x} \tag{5.8}$$

$$\zeta'_\eta(j) := \int_{C_-} \mathrm{d}x \frac{1}{\sinh x} \frac{\partial}{\partial \eta} \frac{\cosh \eta x}{\sinh^j \eta x} \tag{5.9}$$

where C_- is an integral path from $-\infty - \pi \mathrm{i}/2$ to $\infty - \pi \mathrm{i}/2$ and η is equal to ν in case $-1 < \Delta = \cos \pi \nu \leqslant 1$ and is equal to $\phi \mathrm{i}$ in case $1 < \Delta = \cos \pi \phi$. Note that we have used the relations

$$\langle S_j^x S_{j+3}^x \rangle = F\begin{bmatrix} + + + - \\ - + + + \end{bmatrix} + F\begin{bmatrix} + + - - \\ - + - + \end{bmatrix} \tag{5.10}$$

$$\langle S_j^z S_{j+3}^z \rangle = -2F\begin{bmatrix} + - + - \\ + - + - \end{bmatrix} + 2F\begin{bmatrix} + + + + \\ + + + + \end{bmatrix} - 3F\begin{bmatrix} + + \\ + + \end{bmatrix} + \tfrac{3}{4} \tag{5.11}$$

to get the results (5.4) and (5.5). Expressions (5.4) and (5.5) allow us to give the numerical values of the correlation functions with a very high precision. They are plotted in figure 1.

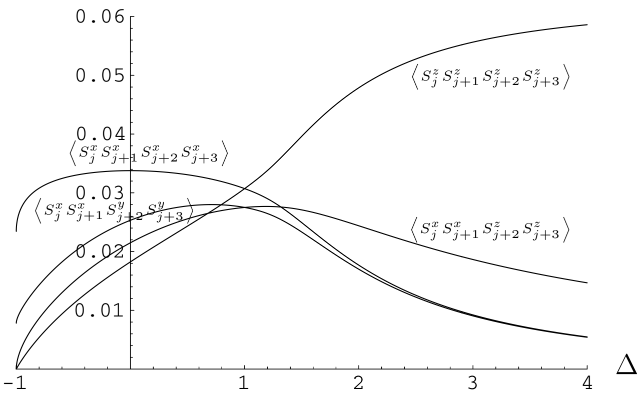


Figure 2. Four-point correlation functions, $\langle S_j^z S_{j+1}^z S_{j+2}^z S_{j+3}^z\rangle$ $\langle S_j^x S_{j+1}^x S_{j+2}^z S_{j+3}^z\rangle$ $\langle S_j^x S_{j+1}^x S_{j+2}^x S_{j+3}^x\rangle$ $\langle S_j^x S_{j+1}^x S_{j+2}^y S_{j+3}^y\rangle$, for the *XXZ* chain.

Table 1. Exact values of the third-neighbour correlation functions at several points of Δ. For comparison, we also give the values for the second-neighbour correlations here.

ν	$\left\langle S_j^z S_{j+2}^z \right\rangle$	$\left\langle S_j^z S_{j+3}^z \right\rangle$	$\left\langle S_j^x S_{j+3}^x \right\rangle$
0	$\frac{1}{12} - \frac{4\log(2)}{3} + \frac{3\zeta(3)}{4}$	$\frac{1}{12} - 3\log(2) + \frac{37\zeta(3)}{6} - \frac{125\zeta(5)}{24} - \frac{14\log(2)\zeta(3)}{3} + \frac{25\log(2)\zeta(5)}{3} - \frac{3\zeta(3)^2}{2}$	$\frac{1}{12} - 3\log(2) + \frac{37\zeta(3)}{6} - \frac{125\zeta(5)}{24} - \frac{14\log(2)\zeta(3)}{3} + \frac{25\log(2)\zeta(5)}{3} - \frac{3\zeta(3)^2}{2}$
$\frac{1}{2}$	0	$-\frac{1}{9\pi^2}$	$-\frac{8}{3\pi^3}$
$\frac{1}{3}$	$\frac{7}{256}$	$-\frac{401}{16\,384}$	$-\frac{4399}{65\,536}$
$\frac{2}{3}$	$\frac{8671}{4096} - \frac{49\sqrt{3}}{64} - \frac{1305}{512\pi}$	$-\frac{25\,162\,841}{4194\,304} + \frac{703\,383\sqrt{3}}{131\,072} + \frac{1018\,791}{262\,144\pi} - \frac{7533\sqrt{3}}{1024\pi} - \frac{19\,683}{4096\pi^2}$	$-\frac{23\,685\,209}{16\,777\,216} + \frac{113\,127\sqrt{3}}{131\,072} + \frac{2138\,535}{1048\,576\pi} - \frac{2673\sqrt{3}}{2048\pi} - \frac{19\,683}{16\,384\pi^2}$
$\frac{1}{4}$	$-\frac{5}{8} + \frac{4}{\pi} - \frac{6}{\pi^2}$	$-\frac{39}{16} + \frac{22}{\pi} - \frac{677}{9\pi^2} + \frac{1088}{9\pi^3} - \frac{256}{3\pi^4}$	$\frac{35\sqrt{2}}{64} - \frac{131\sqrt{2}}{24\pi} + \frac{707\sqrt{2}}{36\pi^2} - \frac{296\sqrt{2}}{9\pi^3} + \frac{64\sqrt{2}}{3\pi^4}$
$\frac{3}{4}$	$\frac{36\,169}{17\,496} - \frac{160\sqrt{3}}{243} - \frac{4}{3\pi} - \frac{608\sqrt{3}}{729\pi} - \frac{2}{3\pi^2}$	$-\frac{2970\,584\,653}{229\,582\,512} + \frac{41\,685\,488\sqrt{3}}{4782\,969} - \frac{22\,418}{531\,441\pi} + \frac{3638\,960\sqrt{3}}{4782\,969\pi} - \frac{2541\,253}{531\,441\pi^2} - \frac{197\,120\sqrt{3}}{19\,683\pi^2} - \frac{1088}{243\pi^3} - \frac{188\,416\sqrt{3}}{59049\pi^3} - \frac{256}{243\pi^4}$	$-\frac{3334\,212\,769\sqrt{2}}{918\,330\,048} + \frac{11\,666\,693\sqrt{6}}{4782\,969} + \frac{235\,397\sqrt{2}}{4251\,528\pi} - \frac{95\,308\sqrt{6}}{4782\,969\pi} - \frac{2738\,083\sqrt{2}}{2125\,764\pi^2} - \frac{52\,544\sqrt{6}}{19\,683\pi^2} - \frac{296\sqrt{2}}{243\pi^3} - \frac{47\,104\sqrt{6}}{59\,049\pi^3} - \frac{64\sqrt{2}}{243\pi^4}$
$\frac{1}{5}$	$-\frac{3529}{512} + \frac{1589\sqrt{5}}{512}$	$-\frac{7128\,183}{32\,768} + \frac{3187\,253\sqrt{5}}{32\,768}$	$\frac{6215\,287}{131\,072} - \frac{2782\,879\sqrt{5}}{131\,072}$
$\frac{1}{6}$	$-\frac{283}{288} + \frac{11\sqrt{3}}{3\pi} - \frac{39}{4\pi^2}$	$-\frac{622\,483}{62\,208} + \frac{5216\sqrt{3}}{81\pi} - \frac{69\,007}{144\pi^2} + \frac{546\sqrt{3}}{\pi^3} - \frac{729}{\pi^4}$	$\frac{202\,393\sqrt{3}}{82\,944} - \frac{27\,697}{576\pi} + \frac{23\,045\sqrt{3}}{192\pi^2} - \frac{1647}{4\pi^3} + \frac{729\sqrt{3}}{4\pi^4}$
1	0	0	$-\frac{1}{8}$

Furthermore, when η is a real rational number, the one-dimensional integrals (5.8) and (5.9) can be integrated analytically and we can get the complete analytical values for (5.4) and (5.5). Some explicit examples are given in table 1.

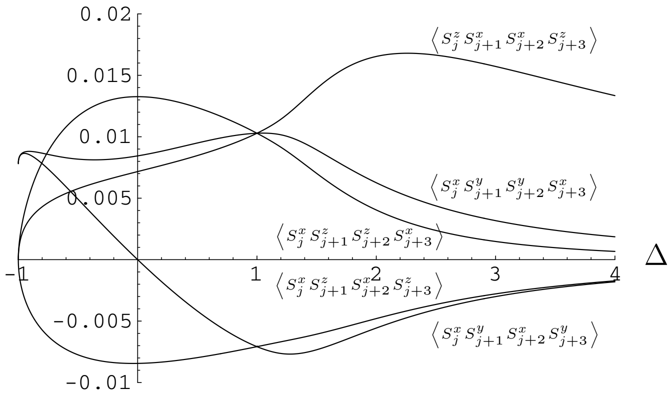


Figure 3. Four-point correlation functions, $\langle S_j^z S_{j+1}^x S_{j+2}^x S_{j+3}^z \rangle$ $\langle S_j^x S_{j+1}^y S_{j+2}^y S_{j+3}^x \rangle$ $\langle S_j^x S_{j+1}^z S_{j+2}^z S_{j+3}^x \rangle$ $\langle S_j^x S_{j+1}^z S_{j+2}^x S_{j+3}^z \rangle$ $\langle S_j^x S_{j+1}^y S_{j+2}^x S_{j+3}^y \rangle$, for the *XXZ* chain.

The polynomial representations of other correlation functions, $\left\langle S_j^z S_{j+1}^z S_{j+2}^z S_{j+3}^z \right\rangle$ etc, can be derived similarly. Here we simply show the shapes of the correlation functions in figures 2 and 3 by using the polynomial representations.

We remark that at $\Delta = 1$ our results reproduce the same values as in [23, 26]. It is immediate to see them from the following asymptotic expansions at $\eta = 0$,

$$\zeta_\eta(1) = -2\log 2\eta^{-1} + \frac{\pi^2}{6}\eta - \frac{\pi^4}{180}\eta^3 + \frac{\pi^6}{945}\eta^5 + \cdots \tag{5.12}$$

$$\zeta_\eta(3) = \frac{3\zeta(3)}{2\pi^2}\eta^{-3} - \frac{\pi^2}{30}\eta + \frac{\pi^4}{189}\eta^3 - \frac{\pi^6}{450}\eta^5 + \cdots \tag{5.13}$$

$$\zeta_\eta(5) = -\frac{15\zeta(5)}{8\pi^4}\eta^{-5} - \frac{\zeta(3)}{2\pi^2}\eta^{-3} + \frac{31\pi^2}{1890}\eta - \frac{41\pi^4}{11\,340}\eta^3 + \frac{31\pi^6}{14\,850}\eta^5 + \cdots \tag{5.14}$$

and so on.

6. Conclusion

We have analysed multiple integral representations of correlation functions for *XXZ* model. In our results, we have given *polynomial representations* for all the adjacent points correlation functions under the restriction that the number of the points is one to four. The *polynomial* means polynomial with respect to specific integrals (1.2) and (1.3) where the coefficients are rational functions of $\sin\pi\eta$ and $\cos\pi\eta$. Here, $\cos\pi\eta$ is equal to Δ, and η is either a real number ($-1 < \Delta \leqslant 1$) or a purely imaginary number ($1 < \Delta$). It is intriguing that the correlation functions are given by the common expressions both in the massless regime and the massive regime. Probably, it will be possible to generalize our results to more general XYZ model.

On the other hand, we can, in principle, study the correlation functions even further for $n \geqslant 5$ using our method. By surveying our results, we conjecture that there exist *polynomial representations* for all these correlation functions, where only the one-dimensional integrals (1.2) and (1.3) with odd integer j appear.

Acknowledgments

The authors are grateful to H E Boos, K Hikami and M Wadati for valuable comments and stimulating discussions. This work is in part supported by Grant-in-Aid for the Scientific Research (B) no 14340099 and also by a 21st Century COE Program at Tokyo Tech'Nanometer-Scale Quantum Physics' from the Ministry of Education, Culture, Sports, Science and Technology. Further GK is supported by the JSPS research fellowships for young scientists and MS is supported by Grant-in Aid for Young Scientists (B) no 14740228.

Appendix A. Canonical form for $F\left[\begin{smallmatrix}+&+&+&+\\+&+&+&+\end{smallmatrix}\right]$

In this section, we derive canonical form of $F\left[\begin{smallmatrix}+&+&+&+\\+&+&+&+\end{smallmatrix}\right]$ as an example. In the following, we assume $F_m^{(n)}$ as a polynomial with respect to y_1, y_2, y_3 and y_4 including negative power terms.

The polynomial T with respect to $F\left[\begin{smallmatrix}+&+&+&+\\+&+&+&+\end{smallmatrix}\right]$ is written as

$$\begin{aligned} T(y_1,y_2,y_3,y_4) &= \frac{(y_1-1)^3(y_2-1)^2(qy_2-1)(y_3-1)(qy_3-1)^2(qy_4-1)^3}{(y_1-qy_2)(y_1-qy_3)(y_1-qy_4)(y_2-qy_3)(y_2-qy_4)(y_3-qy_4)}F_a^{(1)} \\ &= \frac{(y_1-1)^3(y_2-1)^2(qy_2-1)(y_3-1)(qy_3-1)^2(qy_4-1)^3}{(y_1-qy_2)(y_1-qy_3)(y_1-qy_4)(y_2-qy_4)(y_2-qy_3)}F_a^{(2)} \\ &\quad - \frac{(y_1-1)^3(y_2-1)^2(qy_2-1)(y_3-1)(qy_3-1)^2(qy_4-1)^3}{(y_1-qy_2)(y_1-qy_3)(y_1-qy_4)(y_3-qy_4)(y_2-qy_3)}F_a^{(2)} \\ &\quad + \frac{(y_1-1)^3(y_2-1)^2(qy_2-1)(y_3-1)(qy_3-1)^2(qy_4-1)^3}{(y_1-qy_2)(y_1-qy_3)(y_1-qy_4)(y_3-qy_4)(y_2-qy_4)}F_a^{(2)}. \end{aligned} \tag{A.1}$$

Here the second equality is due to an elementary relation

$$\begin{aligned} \frac{1}{(y_k-y_lq)(y_j-y_lq)(y_j-y_kq)} &= \frac{1}{(1-q)y_k}\left[\frac{1}{(y_j-y_lq)(y_j-y_kq)}\right. \\ &\quad \left. - \frac{1}{(y_k-y_lq)(y_j-y_kq)} + \frac{1}{(y_k-y_lq)(y_j-y_lq)}\right]. \end{aligned} \tag{A.2}$$

We named three terms in (A.1) as $I_1^{(2)}$, $I_2^{(2)}$ and $I_3^{(2)}$ order by order. The two terms $I_1^{(2)}$ and $I_3^{(2)}$ are modified as

$$\begin{aligned} I_1^{(2)} &\sim \frac{(y_1-1)^2(y_2-1)^2(y_3-1)(qy_3-1)^2(qy_4-1)^3}{(y_1-qy_2)(y_1-qy_3)(y_1-qy_4)(y_2-qy_4)}F_1^{(3)} \\ &= \frac{(y_1-1)^2(y_2-1)^2(y_3-1)(qy_3-1)^2(qy_4-1)^3}{(y_1-qy_3)(y_1-qy_2)(y_1-qy_4)}F_1^{(4)} \\ &\quad - \frac{(y_1-1)^2(y_2-1)^2(y_3-1)(qy_3-1)^2(qy_4-1)^3}{(y_1-qy_3)(y_1-qy_2)(y_2-qy_4)}F_1^{(4)} \\ &\quad + \frac{(y_1-1)^2(y_2-1)^2(y_3-1)(qy_3-1)^2(qy_4-1)^3}{(y_1-qy_3)(y_1-qy_4)(y_2-qy_4)}F_1^{(4)} \\ &=: I_1^{(4)} + I_2^{(4)} + I_3^{(4)} \end{aligned} \tag{A.3}$$

$$\begin{aligned} I_3^{(2)} &\sim \frac{(y_1-1)^3(y_2-1)(qy_2-1)(y_3-1)(qy_3-1)(qy_4-1)^3}{(y_1-qy_2)(y_1-qy_3)(y_1-qy_4)(y_3-qy_4)}F_3^{(3)} \\ &= \frac{(y_1-1)^3(y_2-1)(qy_2-1)(y_3-1)(qy_3-1)(qy_4-1)^3}{(y_1-qy_2)(y_1-qy_4)(y_3-qy_4)}F_3^{(4)} \end{aligned}$$

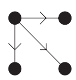
$$
\begin{aligned}
&- \frac{(y_1-1)^3(y_2-1)(qy_2-1)(y_3-1)(qy_3-1)(qy_4-1)^3}{(y_1-qy_2)(y_1-qy_3)(y_3-qy_4)} F_3^{(4)} \\
&+ \frac{(y_1-1)^3(y_2-1)(qy_2-1)(y_3-1)(qy_3-1)(qy_4-1)^3}{(y_1-qy_2)(y_1-qy_3)(y_1-qy_4)} F_3^{(4)} \\
&=: I_{12}^{(4)} + I_{13}^{(4)} + I_{14}^{(4)}.
\end{aligned}
\tag{A.4}
$$

In the two *weak* equalities above, we used relation (2.14). We also used the elementary relation (A.2) in the two equalities above. The term $I_2^{(2)}$ is modified as follows by using elementary relations like (A.2):

$$
\begin{aligned}
I_2^{(2)} = {} & \frac{(y_1-1)^3(y_2-1)^2(qy_2-1)(y_3-1)(qy_3-1)^2(qy_4-1)^3}{(q-1)(y_1-qy_2)(y_1-qy_3)(y_1-qy_4)} F_2^{(3)} \\
&+ \frac{(y_1-1)^3(y_2-1)^2(qy_2-1)(y_3-1)(qy_3-1)^2(qy_4-1)^3}{(q-1)(y_1-qy_3)(y_1-qy_4)(y_2-qy_3)} F_2^{(3)} \\
&+ \frac{(y_1-1)^3(y_2-1)^2(qy_2-1)(y_3-1)(qy_3-1)^2(qy_4-1)^3}{(q^2-1)(y_1-qy_2)(y_1-qy_4)(y_3-qy_4)} F_2^{(3)} \\
&- \frac{(y_1-1)^3(y_2-1)^2(qy_2-1)(y_3-1)(qy_3-1)^2(qy_4-1)^3}{(q^2-1)(y_1-qy_2)(y_1-qy_4)(y_2-qy_3)} F_2^{(3)} \\
&- \frac{(y_1-1)^3(y_2-1)^2(qy_2-1)(y_3-1)(qy_3-1)^2(qy_4-1)^3}{(q-1)(y_1-qy_2)(y_1-qy_3)(y_3-qy_4)} F_2^{(3)} \\
&+ \frac{(y_1-1)^3(y_2-1)^2(qy_2-1)(y_3-1)(qy_3-1)^2(qy_4-1)^3}{(q^2-1)(y_1-qy_2)(y_2-qy_3)(y_3-qy_4)} F_2^{(3)} \\
&- \frac{(y_1-1)^3(y_2-1)^2(qy_2-1)(y_3-1)(qy_3-1)^2(qy_4-1)^3}{(q-1)(y_1-qy_3)(y_2-qy_3)(y_3-qy_4)} F_2^{(3)} \\
&+ \frac{(y_1-1)^3(y_2-1)^2(qy_2-1)(y_3-1)(qy_3-1)^2(qy_4-1)^3 q}{(q^2-1)(y_1-qy_4)(y_2-qy_3)(y_3-qy_4)} F_2^{(3)} \\
=:{} & I_4^{(4)} + I_5^{(4)} + I_6^{(4)} + I_7^{(4)} + I_8^{(4)} + I_9^{(4)} + I_{10}^{(4)} + I_{11}^{(4)}.
\end{aligned}
\tag{A.5}
$$

It is rather troublesome to show the further evaluation directly. Then, we introduce graphs which indicate certain sets of rational functions with respect to $\{y_1, y_2, y_3, y_4\}$ as follows.

Each graph is made from three parts, i.e. circles, directed lines and non-directed lines. First, there are four circles in a graph, which are placed at the four corners. The circle at upper left (right) represents the variable y_1 (y_2), and the circle at lower left (right) represents the variable y_3 (y_4). Next, each graph contains directed lines or non-directed lines which connect two circles and indicates a certain analytic property of rational functions. If a non-directed line connects y_j and y_k, it implies that the order of a pole at $y_j = y_k$ is 1 at most. If y_j and y_k are connected by a directed line from y_j to y_k, the order of a pole at $y_j = qy_k$ is 1 at most, and at the same time the function has a factor $(y_j-1)^{m_j}(qy_k-1)^{m_k}$ with $m_j + m_k \geqslant 4$. Finally, we assume that the rational functions do not have any other pole except for the one at $y_j = 0$. In this way, we can define a set of rational functions corresponding to each graph. For example, $I_4^{(4)}$ is in the set



(A.6)

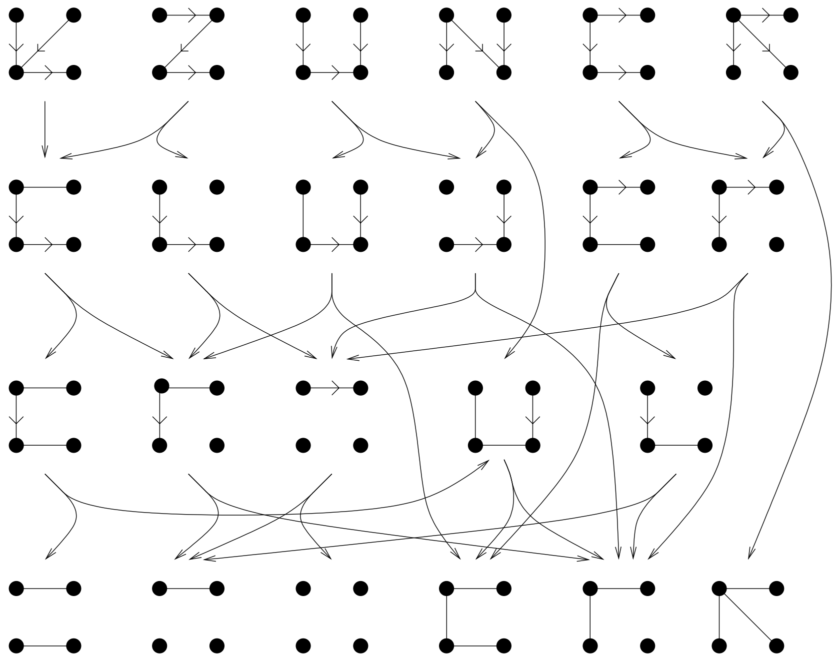


Figure A1. The modification stream of rational functions 1.

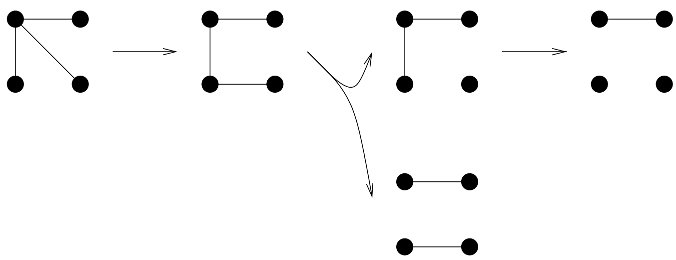


Figure A2. The modification stream of rational functions 2.

We shall show the outline of further evaluation by means of these graphs. In figures A1 and A2, some relations between the graphs are shown,

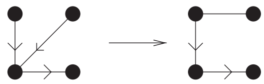
$$\text{`graph1'} \longrightarrow \text{`graph2'} \tag{A.8}$$

indicates that any function in 'graph1' can be modified into a sum of functions in 'graph2' using the week equality. Similarly,

$$\text{`graph1'} \quad \begin{matrix} \nearrow & \text{`graph2'} \\ \searrow & \text{`graph3'} \end{matrix} \tag{A.8}$$

indicates that any function in 'graph1' can be modified into a sum of functions in 'graph2' and 'graph3'. We can modify functions along the arrows between graphs in figures A1 and A2.

Below we prove it for some of the arrows. First, let us consider the upper left arrow in figure A1,

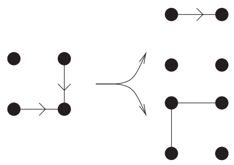


(A.9)

This modification can be proved primitively as follows,

$$
\begin{aligned}
&\frac{(y_1-1)^{m_1}(y_2-1)^{m_2}(y_3-1)^{m_3}(qy_3-1)^{m_4}(qy_4-1)^{m_5}}{(y_1-qy_3)(y_2-qy_3)(y_3-qy_4)}F(y_1,y_2,y_3,y_4)\\
&\quad=\frac{(y_1-1)^{m_1}(y_2-1)^{m_2}(y_3-1)^{m_3}(qy_3-1)^{m_4}(qy_4-1)^{m_5}}{(y_1-y_2)(y_2-qy_3)(y_3-qy_4)}F(y_1,y_2,y_3,y_4)\\
&\qquad+\frac{(y_1-1)^{m_1}(y_2-1)^{m_2}(y_3-1)^{m_3}(qy_3-1)^{m_4}(qy_4-1)^{m_5}}{(y_1-qy_3)(y_2-y_1)(y_3-qy_4)}F(y_1,y_2,y_3,y_4)\\
&\quad\sim\frac{(y_1-1)^{m_1}(y_2-1)^{m_2}(y_3-1)^{m_3}(qy_3-1)^{m_4}(qy_4-1)^{m_5}}{(y_1-y_2)(y_2-qy_3)(y_3-qy_4)}F(y_1,y_2,y_3,y_4)\\
&\qquad-\frac{(y_2-1)^{m_1}(y_1-1)^{m_2}(y_3-1)^{m_3}(qy_3-1)^{m_4}(qy_4-1)^{m_5}}{(y_2-qy_3)(y_1-y_2)(y_3-qy_4)}F(y_2,y_1,y_3,y_4)
\end{aligned}
\tag{A.10}
$$

where F is a polynomial including negative powers. In the weak equality, we used relation (2.12). Next, we prove the relation

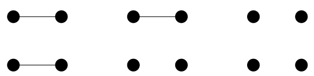


(A.11)

which is one of the arrows in figure A1:

$$
\begin{aligned}
&\frac{(y_2-1)^{m_1}(y_3-1)^{m_2}(qy_4-1)^{m_3}}{(y_2-qy_4)(y_3-qy_4)}F(y_1,y_2,y_3,y_4)\\
&\quad=\frac{(y_2-1)^{m_1}(qy_4-1)^{m_3}}{(y_2-qy_4)}\sum_{j=0}^{m_2-1}(y_3-1)^{m_2-1-j}(qy_4-1)^{j}F(y_1,y_2,y_3,y_4)\\
&\qquad+\frac{(y_2-1)^{m_1}(qy_4-1)^{m_2+m_3}}{(y_2-qy_4)(y_3-qy_4)}F(y_1,y_2,y_3,y_4)\\
&\quad\sim\frac{(y_1-1)^{m_1}(qy_2-1)^{m_3}}{(y_1-qy_2)}\sum_{j=0}^{m_2-1}(y_3-1)^{m_2-1-j}(qy_2-1)^{j}F(y_4,y_1,y_3,y_2)\\
&\qquad+\frac{(y_2-1)^{m_1}(y_1-1)^{m_2+m_3}}{(y_2-y_1)(y_3-y_1)}F(y_4,y_2,y_3,q^{-1}y_1).
\end{aligned}
\tag{A.12}
$$

Here in the weak equality, we used relations (2.12) and (2.13). The other arrows in figure A1 are proved in the same way as case (A.11). As for figure A2, the left arrow is proved by relations like (A.2) and (2.12), the middle arrow by (2.16) and the right arrow by (2.14). Finally, we note that any function in



(A.13)

becomes canonical form using relations (2.18) and (2.20).

Now, let us go back to the modification of (A.1). We note that any of functions $I_1^{(4)}, I_2^{(4)}, \ldots, I_{14}^{(4)}$ is weekly equal to a function in the sets in figure A1. Then the discussion above indicates that $T(y_1, y_2, y_3, y_4)$ can be modified into a canonical form according to the modification streams figures A1 and A2 with relations (2.18) and (2.20). Although we have actually come through the modifications, it is very tedious to show them here. Therefore, we show just the final result,

$$T(y_1, y_2, y_3, y_4) = \frac{(y_1-1)^3(y_2-1)^2(qy_2-1)(y_3-1)(qy_3-1)^2(qy_4-1)^3}{(y_1-qy_2)(y_1-qy_3)(y_1-qy_4)(y_2-qy_3)(y_2-qy_4)(y_3-qy_4)} \sim \frac{A}{(y_2-y_1)(y_4-y_3)} + \frac{B}{(y_2-y_1)} + C \tag{A.14}$$

where

$$\begin{aligned} A = {} & \frac{1-12q+53q^2-42q^3+93q^4+32q^5+27q^6+6q^7+2q^8}{128q^4(1+q)(1+q+q^2)} \\ & - \frac{(1+q)(1+3q+q^2)}{8q^2} y_3 + \frac{(4-9q+45q^2+16q^3+36q^4+21q^5+7q^6)}{16q^2(1+q)(1+q+q^2)} {y_3}^2 \\ & - \frac{(1-3q+10q^2+q^3+6q^4+4q^5+q^6)}{16q^2(1+q+q^2)} {y_3}^3 \\ & + \frac{(3+22q+70q^2+22q^3+3q^4)}{64q^2} y_1 y_3 \\ & - \frac{(1+q)(1+2q+24q^2+2q^3+q^4)}{16q^2} y_1 {y_3}^2 + \frac{(1+q)^2(1+18q^2+q^4)}{64q^2} y_1 {y_3}^3 \\ & + \frac{(2+134q+91q^2+512q^3+413q^4+214q^5+53q^6+20q^7+q^8)}{128q(1+q)(1+q+q^2)} {y_1}^2 {y_3}^2 \\ & - \frac{(7+24q^2+19q^3+6q^4+3q^5+q^6)}{16(1+q+q^2)} {y_1}^2 {y_3}^3 \\ & + \frac{(1+q)^2(3-5q+14q^2-5q^3+3q^4)}{64(1+q+q^2)} {y_1}^3 {y_3}^3, \end{aligned} \tag{A.15}$$

$$\begin{aligned} B = {} & \frac{(6-48q+7q^2+341q^3-101q^4-76q^5-127q^6+125q^7+45q^8)}{64q^3(1+q)(1+q+q^2)} y_4 \\ & - \frac{(2-25q+58q^2+78q^3+33q^4+77q^5-104q^6-14q^7+81q^8+18q^9)}{32q^3(1+q)(1+q+q^2)} {y_4}^2 \\ & + \frac{(1-13q+44q^2+66q^3-38q^4+98q^5+31q^6-51q^7+6q^8+50q^9+10q^{10})}{64q^3(1+q)(1+q+q^2)} {y_4}^3 \\ & + \frac{(-15+92q-67q^2+437q^3-77q^4+141q^5-328q^6+180q^7+121q^8)}{64q^2(1+q)(1+q+q^2)} y_3 {y_4}^2 \\ & - \frac{(-4+47q-41q^2+108q^3-12q^4+76q^5-108q^6+41q^7+41q^8)}{64q^2(1+q+q^2)} y_3 {y_4}^3 \\ & + \frac{(7+q-4q^2+44q^3+5q^4+13q^5-24q^6+14q^7+11q^8+3q^9)}{32q(1+q)(1+q+q^2)} {y_3}^2 {y_4}^3 \end{aligned}$$

$$
\begin{aligned}
&- \frac{(1+q)}{4} y_1 y_4 + \frac{(2-q+2q^2)}{2} y_1 {y_4}^2 - \frac{3(1+q)(1-q+q^2)}{8} y_1 {y_4}^3 \\
&- \frac{5(1+q)(1+q^2)}{16} y_1 y_3 {y_4}^2 + \frac{5(1+q)^2(1-q+q^2)}{32} y_1 y_3 {y_4}^3 \\
&- \frac{(1+q)(1+q^2)(1-q+q^2)}{16} y_1 {y_3}^2 {y_4}^3 \\
&- \frac{(-52+543q^2-199q^3-159q^4-273q^5+182q^6+9q^7)}{64q(1+q)(1+q+q^2)} {y_1}^2 y_4 \\
&+ \frac{(-53+181q-15q^2+96q^3-q^4-429q^5+197q^6+60q^7)}{64q(1+q+q^2)} {y_1}^2 {y_4}^2 \\
&- \frac{(-14+53q+58q^2-81q^3+126q^4-2q^5-148q^6-47q^7+76q^8+15q^9)}{64q(1+q)(1+q+q^2)} {y_1}^2 {y_4}^3 \\
&- \frac{(-13+110q-118q^2+659q^3-55q^4+116q^5-586q^6+161q^7+218q^8)}{64q(1+q)(1+q+q^2)} {y_1}^2 y_3 {y_4}^2 \\
&+ \frac{(-3+55q-62q^2+168q^3+28q^4+100q^5-177q^6+25q^7+70q^8)}{64q(1+q+q^2)} {y_1}^2 y_3 {y_4}^3 \\
&- \frac{(16-q-14q^2+149q^3+49q^4+50q^5-79q^6+18q^7+30q^8+10q^9)}{64(1+q)(1+q+q^2)} {y_1}^2 {y_3}^2 {y_4}^3 \\
&+ \frac{(-13-q+155q^2-40q^3-23q^4-79q^5+53q^6+24q^7)}{64q(1+q+q^2)} {y_1}^3 y_4 \\
&- \frac{(1+q)(-14+51q+9q^2+37q^3+18q^4-117q^5+61q^6+15q^7)}{64q(1+q+q^2)} {y_1}^3 {y_4}^2 \\
&+ \frac{(-2+8q+11q^2-6q^3+22q^4+4q^5-16q^6-4q^7+11q^8+2q^9)}{32q(1+q+q^2)} {y_1}^3 {y_4}^3 \\
&+ \frac{(-4+31q-23q^2+199q^3+18q^4+59q^5-143q^6+47q^7+60q^8)}{64q(1+q+q^2)} {y_1}^3 y_3 {y_4}^2 \\
&- \frac{(1+q)(-1+16q-13q^2+51q^3+16q^4+33q^5-42q^6+8q^7+20q^8)}{64q(1+q+q^2)} {y_1}^3 y_3 {y_4}^3 \\
&+ \frac{(5+q-q^2+45q^3+20q^4+20q^5-17q^6+7q^7+9q^8+3q^9)}{64(1+q+q^2)} {y_1}^3 {y_3}^2 {y_4}^3, \qquad \text{(A.16)}
\end{aligned}
$$

$$
C = \frac{1+3q-16q^2-50q^3+51q^4-59q^5+18q^6-29q^7+29q^8-15q^9-9q^{10}}{64q\,(1+q)\left(1+q+q^2\right)} y_2 {y_3}^2 {y_4}^3. \qquad \text{(A.17)}
$$

Appendix B. Results

We use some notation defined in (5.6)–(5.9).

$$
F\begin{bmatrix} + \\ + \end{bmatrix} = \frac{1}{2} \qquad \text{(B.1)}
$$

$$F\begin{bmatrix} + & + \\ + & + \end{bmatrix} = \frac{1}{2} - \frac{c_1}{2\pi s_1}\zeta_\eta(1) - \frac{1}{2\pi^2}\zeta'_\eta(1) \tag{B.2}$$

$$F\begin{bmatrix} + & - \\ - & + \end{bmatrix} = \frac{1}{2\pi s_1}\zeta_\eta(1) + \frac{c_1}{2\pi^2}\zeta'_\eta(1) \tag{B.3}$$

$$F\begin{bmatrix} + & + & + \\ + & + & + \end{bmatrix} = \frac{1}{2} - \frac{1+2c_2}{2\pi s_2}\zeta_\eta(1) - \frac{3}{4\pi^2}\zeta'_\eta(1) + \frac{3s_1}{8\pi c_1}\zeta_\eta(3) + \frac{1-c_2}{16\pi^2}\zeta'_\eta(3) \tag{B.4}$$

$$F\begin{bmatrix} + & + & - \\ - & + & + \end{bmatrix} = \frac{1}{2\pi s_2}\zeta_\eta(1) + \frac{c_2}{4\pi^2}\zeta'_\eta(1) - \frac{3c_2(1-c_2)}{8\pi s_2}\zeta_\eta(3) - \frac{1-c_2}{16\pi^2}\zeta'_\eta(3) \tag{B.5}$$

$$\begin{aligned}
F\begin{bmatrix} + & + & + & + \\ + & + & + & + \end{bmatrix} &= \frac{1}{2} - \frac{c_2(5+6c_2)}{4\pi c_1 s_3}\zeta_\eta(1) - \frac{11}{12\pi^2}\zeta'_\eta(1) \\
&\quad - \frac{21+61c_2+38c_4+3c_6+c_8}{32\pi c_1 s_3}\zeta_\eta(3) - \frac{5+6c_2+3c_4}{48\pi^2}\zeta'_\eta(3) \\
&\quad - \frac{5(33+61c_2+22c_4+3c_6+c_8)}{64\pi c_1 s_3}\zeta_\eta(5) - \frac{7+2c_2+c_4}{16\pi^2}\zeta'_\eta(5) \\
&\quad + \frac{109+127c_2+58c_4+5c_6+c_8}{32\pi^2 s_1 s_3}\zeta_\eta(1)\zeta_\eta(3) + \frac{c_1(33+16c_2+c_4)}{16\pi^3 s_1}\zeta'_\eta(1)\zeta_\eta(3) \\
&\quad + \frac{c_1(9+c_4)}{16\pi^3 s_1}\zeta_\eta(1)\zeta'_\eta(3) + \frac{35+22c_2+3c_4}{96\pi^4}\zeta'_\eta(1)\zeta'_\eta(3) \\
&\quad + \frac{5(121+163c_2+70c_4+5c_6+c_8)}{64\pi^2 s_1 s_3}\zeta_\eta(1)\zeta_\eta(5) + \frac{5c_1(21+8c_2+c_4)}{16\pi^3 s_1}\zeta'_\eta(1)\zeta_\eta(5) \\
&\quad + \frac{c_1(21+8c_2+c_4)}{16\pi^3 s_1}\zeta_\eta(1)\zeta'_\eta(5) + \frac{35+22c_2+3c_4}{32\pi^4}\zeta'_\eta(1)\zeta'_\eta(5) \\
&\quad - \frac{3(121+163c_2+70c_4+5c_6+c_8)}{128\pi^2 s_1 s_3}\zeta_\eta(3)^2 - \frac{c_1(21+8c_2+c_4)}{16\pi^3 s_1}\zeta_\eta(3)\zeta'_\eta(3) \\
&\quad - \frac{35+22c_2+3c_4}{192\pi^4}\zeta'_\eta(3)^2
\end{aligned} \tag{B.6}$$

$$\begin{aligned}
F\begin{bmatrix} + & - & + & - \\ + & - & + & - \end{bmatrix} &= \frac{c_2}{4\pi c_1 s_3}\zeta_\eta(1) + \frac{1}{12\pi^2}\zeta'_\eta(1) - \frac{24+23c_2+14c_4+c_6}{16\pi c_1 s_3}\zeta_\eta(3) \\
&\quad - \frac{4+3c_2}{24\pi^2}\zeta'_\eta(3) - \frac{5(20+27c_2+12c_4+c_6)}{32\pi c_1 s_3}\zeta_\eta(5) - \frac{3+2c_2}{8\pi^2}\zeta'_\eta(5) \\
&\quad + \frac{3(4+c_2)(3+6c_2+c_4)}{16\pi^2 s_1 s_3}\zeta_\eta(1)\zeta_\eta(3) + \frac{5c_1(4+c_2)}{8\pi^3 s_1}\zeta'_\eta(1)\zeta_\eta(3) \\
&\quad + \frac{c_1(4+c_2)}{8\pi^3 s_1}\zeta_\eta(1)\zeta'_\eta(3) + \frac{19+11c_2}{48\pi^4}\zeta'_\eta(1)\zeta'_\eta(3) \\
&\quad + \frac{15(38+63c_2+18c_4+c_6)}{64\pi^2 s_1 s_3}\zeta_\eta(1)\zeta_\eta(5) + \frac{15c_1(4+c_2)}{8\pi^3 s_1}\zeta'_\eta(1)\zeta_\eta(5) \\
&\quad + \frac{3c_1(4+c_2)}{8\pi^3 s_1}\zeta_\eta(1)\zeta'_\eta(5) + \frac{19+11c_2}{16\pi^4}\zeta'_\eta(1)\zeta'_\eta(5)
\end{aligned}$$

$$
- \frac{9(38 + 63c_2 + 18c_4 + c_6)}{128\pi^2 s_1 s_3} \zeta_\eta(3)^2 - \frac{3c_1(4 + c_2)}{8\pi^3 s_1} \zeta_\eta(3)\zeta'_\eta(3)
- \frac{19 + 11c_2}{96\pi^4} \zeta'_\eta(3)^2 \tag{B.7}
$$

$$
\begin{aligned}
F\begin{bmatrix} + & + & + & - \\ + & + & - & + \end{bmatrix} &= \frac{c_2}{2\pi s_3}\zeta_\eta(1) + \frac{c_1}{6\pi^2}\zeta'_\eta(1) + \frac{11 + 41c_2 + 9c_4 + c_6}{16\pi s_3}\zeta_\eta(3) \\
&+ \frac{c_1(4 + 3c_2)}{24\pi^2}\zeta'_\eta(3) + \frac{5(17 + 33c_2 + 9c_4 + c_6)}{32\pi s_3}\zeta_\eta(5) + \frac{c_1(7 + 3c_2)}{16\pi^2}\zeta'_\eta(5) \\
&- \frac{(4 + c_2)(20 + 27c_2 + 12c_4 + c_6)}{16\pi^2 s_2 s_3}\zeta_\eta(1)\zeta_\eta(3) - \frac{49 + 48c_2 + 3c_4}{32\pi^3 s_1}\zeta'_\eta(1)\zeta_\eta(3) \\
&- \frac{11 + 8c_2 + c_4}{32\pi^3 s_1}\zeta_\eta(1)\zeta'_\eta(3) - \frac{c_1(11 + 4c_2)}{24\pi^4}\zeta'_\eta(1)\zeta'_\eta(3) \\
&- \frac{5(219 + 328c_2 + 148c_4 + 24c_6 + c_8)}{64\pi^2 s_2 s_3}\zeta_\eta(1)\zeta_\eta(5) \\
&- \frac{5(15 + 14c_2 + c_4)}{16\pi^3 s_1}\zeta'_\eta(1)\zeta_\eta(5) - \frac{15 + 14c_2 + c_4}{16\pi^3 s_1}\zeta_\eta(1)\zeta'_\eta(5) \\
&- \frac{c_1(11 + 4c_2)}{8\pi^4}\zeta'_\eta(1)\zeta'_\eta(5) + \frac{3(219 + 328c_2 + 148c_4 + 24c_6 + c_8)}{128\pi^2 s_2 s_3}\zeta_\eta(3)^2 \\
&+ \frac{15 + 14c_2 + c_4}{16\pi^3 s_1}\zeta_\eta(3)\zeta'_\eta(3) + \frac{c_1(11 + 4c_2)}{48\pi^4}\zeta'_\eta(3)^2
\end{aligned} \tag{B.8}
$$

$$
\begin{aligned}
F\begin{bmatrix} + & + & + & - \\ + & - & + & + \end{bmatrix} &= \frac{c_1}{2\pi s_3}\zeta_\eta(1) + \frac{c_2}{6\pi^2}\zeta'_\eta(1) + \frac{c_1(19 + 3c_2 + 9c_4)}{8\pi s_3}\zeta_\eta(3) \\
&+ \frac{3 + 11c_2}{48\pi^2}\zeta'_\eta(3) + \frac{15c_1(2 + 2c_2 + c_4)}{8\pi s_3}\zeta_\eta(5) + \frac{c_1{}^2(4 + c_2)}{8\pi^2}\zeta'_\eta(5) \\
&- \frac{3(28 + 51c_2 + 18c_4 + 3c_6)}{32\pi^2 s_1 s_3}\zeta_\eta(1)\zeta_\eta(3) - \frac{c_1(9 + c_2)(3 + 2c_2)}{16\pi^3 s_1}\zeta'_\eta(1)\zeta_\eta(3) \\
&- \frac{5c_1{}^3}{8\pi^3 s_1}\zeta_\eta(1)\zeta'_\eta(3) - \frac{5c_1{}^2(5 + c_2)}{48\pi^4}\zeta'_\eta(1)\zeta'_\eta(3) \\
&- \frac{5(106 + 177c_2 + 66c_4 + 11c_6)}{64\pi^2 s_1 s_3}\zeta_\eta(1)\zeta_\eta(5) - \frac{5c_1(33 + 26c_2 + c_4)}{32\pi^3 s_1}\zeta'_\eta(1)\zeta_\eta(5) \\
&- \frac{c_1(33 + 26c_2 + c_4)}{32\pi^3 s_1}\zeta_\eta(1)\zeta'_\eta(5) - \frac{5c_1{}^2(5 + c_2)}{16\pi^4}\zeta'_\eta(1)\zeta'_\eta(5) \\
&+ \frac{3(106 + 177c_2 + 66c_4 + 11c_6)}{128\pi^2 s_1 s_3}\zeta_\eta(3)^2 + \frac{c_1(33 + 26c_2 + c_4)}{32\pi^3 s_1}\zeta_\eta(3)\zeta'_\eta(3) \\
&+ \frac{5c_1{}^2(5 + c_2)}{96\pi^4}\zeta'_\eta(3)^2
\end{aligned} \tag{B.9}
$$

$$F\begin{bmatrix} + & + & - & + \\ + & - & + & + \end{bmatrix} = \frac{1}{2\pi s_3}\zeta_\eta(1) + \frac{c_1}{6\pi^2}\zeta_\eta'(1) + \frac{19+5c_2+7c_4}{8\pi s_3}\zeta_\eta(3)$$
$$+\frac{7c_1}{24\pi^2}\zeta_\eta'(3) + \frac{15(2+2c_2+c_4)}{8\pi s_3}\zeta_\eta(5) + \frac{c_1(4+c_2)}{8\pi^2}\zeta_\eta'(5)$$
$$-\frac{3(28+51c_2+18c_4+3c_6)}{16\pi^2 s_2 s_3}\zeta_\eta(1)\zeta_\eta(3) - \frac{(9+c_2)(3+2c_2)}{16\pi^3 s_1}\zeta_\eta'(1)\zeta_\eta(3)$$
$$-\frac{5{c_1}^2}{8\pi^3 s_1}\zeta_\eta(1)\zeta_\eta'(3) - \frac{5c_1(5+c_2)}{48\pi^4}\zeta_\eta'(1)\zeta_\eta'(3)$$
$$-\frac{5(106+177c_2+66c_4+11c_6)}{32\pi^2 s_2 s_3}\zeta_\eta(1)\zeta_\eta(5) - \frac{5(33+26c_2+c_4)}{32\pi^3 s_1}\zeta_\eta'(1)\zeta_\eta(5)$$
$$-\frac{33+26c_2+c_4}{32\pi^3 s_1}\zeta_\eta(1)\zeta_\eta'(5) - \frac{5c_1(5+c_2)}{16\pi^4}\zeta_\eta'(1)\zeta_\eta'(5)$$
$$+\frac{3(106+177c_2+66c_4+11c_6)}{64\pi^2 s_2 s_3}\zeta_\eta(3)^2 + \frac{33+26c_2+c_4}{32\pi^3 s_1}\zeta_\eta(3)\zeta_\eta'(3)$$
$$+\frac{5c_1(5+c_2)}{96\pi^4}\zeta_\eta'(3)^2 \tag{B.10}$$

$$F\begin{bmatrix} + & + & + & - \\ - & + & + & + \end{bmatrix} = \frac{1}{2\pi s_3}\zeta_\eta(1) - \frac{c_1(1-2c_2)}{6\pi^2}\zeta_\eta'(1) + \frac{15+7c_2+5c_4+4c_6}{8\pi s_3}\zeta_\eta(3)$$
$$-\frac{c_1(1-8c_2)}{24\pi^2}\zeta_\eta'(3) + \frac{5(5+6c_2+3c_4+c_6)}{8\pi s_3}\zeta_\eta(5) + \frac{c_1(5+4c_2+c_4)}{16\pi^2}\zeta_\eta'(5)$$
$$-\frac{c_1(14+48c_2+9c_4+4c_6)}{8\pi^2 s_1 s_3}\zeta_\eta(1)\zeta_\eta(3) - \frac{34+53c_2+12c_4+c_6}{32\pi^3 s_1}\zeta_\eta'(1)\zeta_\eta(3)$$
$$-\frac{5+3c_2+2c_4}{16\pi^3 s_1}\zeta_\eta(1)\zeta_\eta'(3) - \frac{{c_1}^3(11+4c_2)}{24\pi^4}\zeta_\eta'(1)\zeta_\eta'(3)$$
$$-\frac{5c_1(23+48c_2+15c_4+4c_6)}{16\pi^2 s_1 s_3}\zeta_\eta(1)\zeta_\eta(5) - \frac{5{c_1}^2(15+14c_2+c_4)}{16\pi^3 s_1}\zeta_\eta'(1)\zeta_\eta(5)$$
$$-\frac{{c_1}^2(15+14c_2+c_4)}{16\pi^3 s_1}\zeta_\eta(1)\zeta_\eta'(5) - \frac{{c_1}^3(11+4c_2)}{8\pi^4}\zeta_\eta'(1)\zeta_\eta'(5)$$
$$+\frac{3c_1(23+48c_2+15c_4+4c_6)}{32\pi^2 s_1 s_3}\zeta_\eta(3)^2 + \frac{{c_1}^2(15+14c_2+c_4)}{16\pi^3 s_1}\zeta_\eta(3)\zeta_\eta'(3)$$
$$+\frac{{c_1}^3(11+4c_2)}{48\pi^4}\zeta_\eta'(3)^2 \tag{B.11}$$

$$F\begin{bmatrix} + & + & - & - \\ - & + & - & + \end{bmatrix} = \frac{1}{4\pi s_3}\zeta_\eta(1) - \frac{c_1(1-2c_2)}{12\pi^2}\zeta_\eta'(1) - \frac{c_2(19+12c_2)}{8\pi s_3}\zeta_\eta(3)$$
$$-\frac{c_1(5+2c_2)}{24\pi^2}\zeta_\eta'(3) - \frac{5(8+15c_2+6c_4+c_6)}{16\pi s_3}\zeta_\eta(5) - \frac{c_1(2+3c_2)}{8\pi^2}\zeta_\eta'(5)$$
$$+\frac{c_1(56+57c_2+36c_4+c_6)}{16\pi^2 s_1 s_3}\zeta_\eta(1)\zeta_\eta(3) + \frac{43+46c_2+11c_4}{32\pi^3 s_1}\zeta_\eta'(1)\zeta_\eta(3)$$

$$+\frac{5+14c_2+c_4}{32\pi^3 s_1}\zeta_\eta(1)\zeta'_\eta(3)+\frac{{c_1}^3(14+c_2)}{24\pi^4}\zeta'_\eta(1)\zeta'_\eta(3)$$
$$+\frac{15c_1(20+27c_2+12c_4+c_6)}{32\pi^2 s_1 s_3}\zeta_\eta(1)\zeta_\eta(5)+\frac{15{c_1}^2(3+2c_2)}{8\pi^3 s_1}\zeta'_\eta(1)\zeta_\eta(5)$$
$$+\frac{3{c_1}^2(3+2c_2)}{8\pi^3 s_1}\zeta_\eta(1)\zeta'_\eta(5)+\frac{{c_1}^3(14+c_2)}{8\pi^4}\zeta'_\eta(1)\zeta'_\eta(5)$$
$$-\frac{9c_1(20+27c_2+12c_4+c_6)}{64\pi^2 s_1 s_3}\zeta_\eta(3)^2-\frac{3{c_1}^2(3+2c_2)}{8\pi^3 s_1}\zeta_\eta(3)\zeta'_\eta(3)$$
$$-\frac{{c_1}^3(14+c_2)}{48\pi^4}\zeta'_\eta(3)^2 \qquad \text{(B.12)}$$

$$F\begin{bmatrix} + & + & - & - \\ - & - & + & + \end{bmatrix}=\frac{1}{4\pi c_1 s_3}\zeta_\eta(1)+\frac{c_4}{12\pi^2}\zeta'_\eta(1)-\frac{6+15c_2+10c_4}{8\pi c_1 s_3}\zeta_\eta(3)$$
$$-\frac{6+c_4}{24\pi^2}\zeta'_\eta(3)-\frac{5c_2(11+15c_2+3c_4+c_6)}{16\pi c_1 s_3}\zeta_\eta(5)-\frac{2+2c_2+c_4}{8\pi^2}\zeta'_\eta(5)$$
$$+\frac{46+57c_2+36c_4+11c_6}{16\pi^2 s_1 s_3}\zeta_\eta(1)\zeta_\eta(3)+\frac{c_1(12+9c_2+4c_4)}{8\pi^3 s_1}\zeta'_\eta(1)\zeta_\eta(3)$$
$$+\frac{5c_1c_2}{8\pi^3 s_1}\zeta_\eta(1)\zeta'_\eta(3)+\frac{{c_1}^2(13+16c_2+c_4)}{48\pi^4}\zeta'_\eta(1)\zeta'_\eta(3)$$
$$+\frac{5(51+76c_2+40c_4+12c_6+c_8)}{32\pi^2 s_1 s_3}\zeta_\eta(1)\zeta_\eta(5)+\frac{5c_1(6+7c_2+2c_4)}{8\pi^3 s_1}\zeta'_\eta(1)\zeta_\eta(5)$$
$$+\frac{c_1(6+7c_2+2c_4)}{8\pi^3 s_1}\zeta_\eta(1)\zeta'_\eta(5)+\frac{{c_1}^2(13+16c_2+c_4)}{16\pi^4}\zeta'_\eta(1)\zeta'_\eta(5)$$
$$-\frac{3(51+76c_2+40c_4+12c_6+c_8)}{64\pi^2 s_1 s_3}\zeta_\eta(3)^2-\frac{c_1(6+7c_2+2c_4)}{8\pi^3 s_1}\zeta_\eta(3)\zeta'_\eta(3)$$
$$-\frac{{c_1}^2(13+16c_2+c_4)}{96\pi^4}\zeta'_\eta(3)^2 \qquad \text{(B.13)}$$

$$F\begin{bmatrix} + & - & + & - \\ - & + & - & + \end{bmatrix}=\frac{1}{4\pi c_1 s_3}\zeta_\eta(1)+\frac{c_2}{12\pi^2}\zeta'_\eta(1)-\frac{11+37c_2+13c_4+c_6}{16\pi c_1 s_3}\zeta_\eta(3)$$
$$-\frac{3+4c_2}{24\pi^2}\zeta'_\eta(3)-\frac{5(8+15c_2+6c_4+c_6)}{16\pi c_1 s_3}\zeta_\eta(5)-\frac{2+3c_2}{8\pi^2}\zeta'_\eta(5)$$
$$+\frac{55+60c_2+33c_4+2c_6}{16\pi^2 s_1 s_3}\zeta_\eta(1)\zeta_\eta(3)+\frac{5c_1(3+2c_2)}{8\pi^3 s_1}\zeta'_\eta(1)\zeta_\eta(3)$$
$$+\frac{c_1(3+2c_2)}{8\pi^3 s_1}\zeta_\eta(1)\zeta'_\eta(3)+\frac{{c_1}^2(14+c_2)}{24\pi^4}\zeta'_\eta(1)\zeta'_\eta(3)$$
$$+\frac{15(20+27c_2+12c_4+c_6)}{32\pi^2 s_1 s_3}\zeta_\eta(1)\zeta_\eta(5)+\frac{15c_1(3+2c_2)}{8\pi^3 s_1}\zeta'_\eta(1)\zeta_\eta(5)$$
$$+\frac{3c_1(3+2c_2)}{8\pi^3 s_1}\zeta_\eta(1)\zeta'_\eta(5)+\frac{{c_1}^2(14+c_2)}{8\pi^4}\zeta'_\eta(1)\zeta'_\eta(5)$$

$$
- \frac{9(20 + 27c_2 + 12c_4 + c_6)}{64\pi^2 s_1 s_3} \zeta_\eta(3)^2 - \frac{3c_1(3 + 2c_2)}{8\pi^3 s_1} \zeta_\eta(3)\zeta'_\eta(3)
$$

$$
- \frac{{c_1}^2(14 + c_2)}{48\pi^4} \zeta'_\eta(3)^2 \tag{B.14}
$$

$$
F\begin{bmatrix} + & - & - & + \\ - & + & + & - \end{bmatrix} = \frac{1}{4\pi c_1 s_3}\zeta_\eta(1) + \frac{1}{12\pi^2}\zeta'_\eta(1) - \frac{5 + 18c_2 + 7c_4 + c_6}{8\pi c_1 s_3}\zeta_\eta(3)
$$

$$
- \frac{1 + 6c_2}{24\pi^2}\zeta'_\eta(3) - \frac{5(8 + 15c_2 + 6c_4 + c_6)}{16\pi c_1 s_3}\zeta_\eta(5) - \frac{2 + 3c_2}{8\pi^2}\zeta'_\eta(5)
$$

$$
+ \frac{55 + 60c_2 + 33c_4 + 2c_6}{16\pi^2 s_1 s_3}\zeta_\eta(1)\zeta_\eta(3) + \frac{5c_1(3 + 2c_2)}{8\pi^3 s_1}\zeta'_\eta(1)\zeta_\eta(3)
$$

$$
+ \frac{c_1(3 + 2c_2)}{8\pi^3 s_1}\zeta_\eta(1)\zeta'_\eta(3) + \frac{{c_1}^2(14 + c_2)}{24\pi^4}\zeta'_\eta(1)\zeta'_\eta(3)
$$

$$
+ \frac{15(20 + 27c_2 + 12c_4 + c_6)}{32\pi^2 s_1 s_3}\zeta_\eta(1)\zeta_\eta(5) + \frac{15c_1(3 + 2c_2)}{8\pi^3 s_1}\zeta'_\eta(1)\zeta_\eta(5)
$$

$$
+ \frac{3c_1(3 + 2c_2)}{8\pi^3 s_1}\zeta_\eta(1)\zeta'_\eta(5) + \frac{{c_1}^2(14 + c_2)}{8\pi^4}\zeta'_\eta(1)\zeta'_\eta(5)
$$

$$
- \frac{9(20 + 27c_2 + 12c_4 + c_6)}{64\pi^2 s_1 s_3}\zeta_\eta(3)^2 - \frac{3c_1(3 + 2c_2)}{8\pi^3 s_1}\zeta_\eta(3)\zeta'_\eta(3)
$$

$$
- \frac{{c_1}^2(14 + c_2)}{48\pi^4}\zeta'_\eta(3)^2. \tag{B.15}
$$

References

[1] Bethe H A 1931 *Z. Phys.* **71** 205
[2] Korepin V E, Bogoliubov N M and Izergin A G 1993 *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge: Cambridge University Press)
[3] Takahashi M 1999 *Thermodynamics of One-Dimensional Solvable Models* (Cambridge: Cambridge University Press)
[4] Lieb E, Schultz T and Mattis D 1961 *Ann. Phys.* **16** 407
[5] McCoy B M 1968 *Phys. Rev.* **173** 531
[6] Shiroishi M, Takahashi M and Nishiyama Y 2001 *J. Phys. Soc. Japan* **70** 3535
[7] Kitanine N, Maillet J M, Slavnov N A and Terras V 2002 *Nucl. Phys.* B **642** 433
[8] Korepin V E, Izergin A G, Essler F H L and Uglov D B 1994 *Phys. Lett.* A **190** 182
[9] Luther A and Peschel I 1975 *Phys. Rev.* B **12** 3908
[10] Lukyanov S and Zamolodchikov A 1997 *Nucl. Phys.* B **493** 571
[11] Lukyanov S 1999 *Phys. Rev.* B **59** 11163
[12] Lukyanov S and Terras V 2003 *Nucl. Phys.* B **654** 323
[13] Kitanine N, Maillet J M, Slavnov N A and Terras V 2002 *J. Phys. A: Math. Gen.* **35** L753
[14] Korepin V E, Lukyanov S, Nishiyama Y and Shiroishi M 2003 *Phys. Let.* A **312** 21
[15] Kitanine N, Maillet J M, Slavnov N A and Terras V 2002 *J. Phys. A: Math. Gen.* **35** L385
[16] Kitanine N, Maillet J M, Slavnov N A and Terras V 2002 *Nucl. Phys.* B **641** 487
[17] Kitanine N, Maillet J M and Terras V 2000 *Nucl. Phys.* B **567** 554
[18] Jimbo M, Miki K, Miwa T and Nakayashiki A 1992 *Phys. lett.* A **168** 256
[19] Jimbo M and Miwa T 1995 *Algebraic Analysis of Solvable Lattice Models* (Providence, RI: American Mathematical Society)
[20] Nakayashiki A 1994 *Int. J. Mod. Phys.* A **9** 5673
[21] Jimbo M and Miwa T 1996 *J. Phys. A: Math. Gen.* **29** 2923
[22] Boos H E and Korepin V E 2001 *J. Phys. A: Math. Gen.* **34** 5311

[23] Boos H E and Korepin V E 2002 *Integrable models and Beyond* ed M Kashiwara and T Miwa (Boston: Birkhäuser) *Preprint* hep-th/0105144
[24] Boos H E, Korepin V E, Nishiyama Y and Shiroishi M 2002 *J. Phys. A: Math. Gen.* **35** 4443
[25] Boos H E, Korepin V E and Smirnov F A 2003 *Nucl. Phys.* B **658** 417
[26] Sakai K, Shiroishi M, Nishiyama Y and Takahashi M 2003 *Phys. Rev.* E **67** 065101
[27] Kato G, Shiroishi M, Takahashi M and Sakai K 2003 *J. Phys. A: Math. Gen.* **36** L337
[28] Takahashi M, Kato G and Shiroishi M 2004 *J. Phys. Soc. Japan* **73** 245
[29] Boos H E, Korepin V E and Smirnov F A 2003 *Preprint* hep-th/0305135 (Moscow *Math. J.* at press)